

Methods of infinite dimensional Morse theory for geodesics on Finsler manifolds

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Abstract

Corresponding to Gromoll-Meyer's Morse theory for geodesics on Riemannian manifolds we develop a similar theory on Finsler manifolds. To be more precise, we obtain the shifting theorems of the critical groups of critical points and critical orbits for the energy functionals of Finsler metrics on the natural Hilbert manifolds on which the Palais-Smale condition holds, and also prove two results on critical groups of iterated closed geodesics whose corresponding versions on Riemannian manifolds must be proved with the splitting lemma. Our approach is based on the splitting lemma for nonsmooth functionals that the author recently developed in [32, 33, 34], and involves neither finite-dimensional approximations nor any Palais' results in [41]. As a simple application we give a generalization version on Finsler manifolds for Theorem 3 in [V. Bangert and W. Klingenberg, Homology generated by iterated closed geodesics, Topology, 22(1983),379-388].

Keywords: Finsler metric; Geodesics; Morse theory; Critical groups; Splitting theorem; Shifting theorem

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1 Introduction and results

Let M be a smooth manifold of dimension n . For an integer $k \geq 2$ or $k = \infty$, a C^k **Finsler metric** on M is a continuous function $F : TM \rightarrow \mathbb{R}$ satisfying the following properties:

- (i) It is C^k and positive in $TM \setminus \{0\}$.
- (ii) $F(x, ty) = tF(x, y)$ for every $t > 0$ and $(x, y) \in TM$.
- (iii) $L := F^2$ is fiberwise strongly convex, that is, for any $(x, y) \in TM \setminus \{0\}$ the symmetric bilinear form (the fiberwise Hessian operator)

$$g(x, y) : T_x M \times T_x M \rightarrow \mathbb{R}, (u, v) \mapsto \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [L(x, y + su + tv)] \Big|_{s=t=0}$$

is positive definite.

Because of the positive homogeneity it is easily checked that $g(x, \lambda y) = g(x, y)$ for any $(x, y) \in TM \setminus \{0\}$ and $\lambda > 0$. Euler theorem implies $g(x, y)[y, y] = (F(x, y))^2 = L(x, y)$. A smooth manifold M endowed with a C^k Finsler metric is called a C^k **Finsler manifold**. Note that $L = F^2$ is of class C^{2-0} , and of class C^2 if and only if it is a norm of a Riemannian metric. A Finsler metric F is said to be **reversible** if $F(x, -v) = F(x, v)$ for all $(x, v) \in TM$. We also say a Finsler metric to **dominate** a Riemannian metric h on M if $F(x, v) \geq C_0 \sqrt{h_x(v, v)}$ for some $C_0 > 0$ and all $(x, v) \in TM$. (Clearly, every Finsler metric on a compact manifold dominates a Riemannian metric). The length of a Lipschitz continuous curve $\gamma : [a, b] \rightarrow M$ with respect to the Finsler structure F is defined by $l_F(\gamma) = \int_a^b F(\gamma(t), \dot{\gamma}(t)) dt$. However for a non-reversible F it only induces a non-symmetric distance and hence leads to the notions of forward and backward compete (and geodesically complete). A differentiable $\gamma = \gamma(t)$ is said to have **constant speed** if $F(\gamma(t), \dot{\gamma}(t))$ is constant along γ . According to Proposition 5.1.1 on the page 115 of [4] or [44], a regular piecewise C^k curve $\gamma : [a, b] \rightarrow M$ is called a F -**geodesic** if for any piecewise C^k variations of it that keep its end-points fixed, $H : (-\epsilon, \epsilon) \times [a, b] \rightarrow M$ (with $H(0, \cdot) = \gamma$), it holds that $\frac{d}{ds} l_F(H(s, \cdot))|_{s=0} = 0$. When $k \geq 4$, in terms of Chern connection D a constant speed geodesic is characterized by the condition $D_{\dot{\gamma}} \dot{\gamma} = 0$, where $D_{\dot{\gamma}}$ is the covariant derivative along γ with respect to D .

Let (M, F) be a n -dimensional C^k Finsler manifold with a complete Riemannian metric h , and let $N \subset M \times M$ be a smooth submanifold. Denote by $I = [0, 1]$ the unit interval, and by $W^{1,2}(I, M)$ the space of absolutely continuous curves $\gamma : I \rightarrow M$ such that $\int_0^1 \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt < \infty$, where $\langle u, v \rangle = h_x(u, v)$ for $u, v \in T_x M$. Then $W^{1,2}(I, M) \subset C^0(I, M)$. Set

$$\Lambda_N(M) := \{\gamma \in W^{1,2}(I, M) \mid (\gamma(0), \gamma(1)) \in N\}.$$

It is a Riemannian-Hilbert submanifold of $W^{1,2}(I, M)$ of codimension $\text{codim}(N)$. If $\gamma \in C^2(I, M) \cap \Lambda_N(M)$ then the pull-back bundle $\gamma^* TM \rightarrow I$ is a C^1 vector bundle. Let $W_N^{1,2}(\gamma^* TM)$ be the set of absolutely continuous sections $\xi : I \rightarrow \gamma^* TM$ such that

$$(\xi(0), \xi(1)) \in T_{(\gamma(0), \gamma(1))} N \quad \text{and} \quad \int_0^1 \langle \nabla_{\dot{\gamma}}^h \xi(t), \nabla_{\dot{\gamma}}^h \xi(t) \rangle dt < \infty.$$

Here $\nabla_{\dot{\gamma}}^h$ is the covariant derivative along γ with respect to the Levi-Civita connection of the metric h . Then $T_{\gamma} \Lambda_N(M) = W_N^{1,2}(\gamma^* TM)$ is equipped with the scalar product

$$\langle \xi, \eta \rangle_1 = \int_0^1 \langle \xi(t), \eta(t) \rangle dt + \int_0^1 \langle \nabla_{\dot{\gamma}}^h \xi(t), \nabla_{\dot{\gamma}}^h \eta(t) \rangle dt. \quad (1.1)$$

For a general $\gamma \in \Lambda_N(M) \setminus C^2(I, M)$, note that $\gamma^* TM \rightarrow I$ is only a bundle of class H^1 , that is, it admits a system of local trivializations whose transition maps are of class $W^{1,2}$. Fortunately, since $W^{1,2}(I, \mathbb{R}^N)$ is a Banach algebra, one can still define $W^{1,2}$ -section of $\gamma^* TM$, and the set $W^{1,2}(\gamma^* TM)$ of such sections is also a well-defined Hilbert space with the inner product given by (1.1) (using the L^2 covariant derivative

along γ associated to the Levi-Civita connection of the metric h). Hence we have also $T_\gamma \Lambda_N(M) = W_N^{1,2}(\gamma^* TM)$ in this case. For a detailed argument of these, see Appendix C of [37]. Consider the energy functional on (M, F) ,

$$\mathcal{L} : \Lambda_N(M) \rightarrow \mathbb{R}, \gamma \mapsto \int_0^1 L(\gamma(t), \dot{\gamma}(t)) dt = \int_0^1 [F(\gamma(t), \dot{\gamma}(t))]^2 dt. \quad (1.2)$$

Proposition 1.1 (i) *The functional \mathcal{L} is C^{2-} , i.e. \mathcal{L} is C^1 and its differential $d\mathcal{L}$ is locally Lipschitz.*

(ii) *A curve $\gamma \in \Lambda_N(M)$ is a constant (non-zero) speed F -geodesic satisfying the boundary condition*

$$g^F(\gamma(0), \dot{\gamma}(0))[u, \dot{\gamma}(0)] = g^F(\gamma(1), \dot{\gamma}(1))[v, \dot{\gamma}(1)] \quad \forall (u, v) \in T_{(\gamma(0), \gamma(1))} N$$

if and only if it is a (non constant) critical point of \mathcal{L} .

(iii) *Suppose that (M, F) is forward (resp. backward) complete and that N is a closed submanifold of $M \times M$ such that the first projection (resp. the second projection) of N to M is compact. Then \mathcal{L} satisfies the Palais-Smale condition on $\Lambda_N(M)$.*

When M is compact and $N = \Delta_M$ (diagonal) this proposition was proved in [38] by Mercuri. Kozma, Kristály and Varga [28] proved (i), (ii) if N is a product of two submanifolds M_1 and M_2 in M , and (iii) if F dominates a complete Riemannian metric on M and $N \subset M \times M$ is a closed submanifold of $M \times M$ such that $P_1(N) \subset M$ or $P_2(N) \subset M$ is compact. The above version were obtained by Caponio, Javaloyes and Masiello [13] recently. A curve $\gamma \in \Lambda_N(M)$ is called **regular** if $\dot{\gamma} \neq 0$ a.e. in $[0, 1]$. When $N = \{p, q\}$ for some $p, q \in M$, Caponio proved in [10, Prop.3.2]: if \mathcal{L} is twice differentiable at a regular curve $\gamma \in \Lambda_N(M)$ then for a.e. $s \in [0, 1]$, the function

$$v \in T_{\gamma(s)} M \rightarrow F^2(\gamma(s), v)$$

is a quadratic positive definite form. This suggests that the results in [39, 22] cannot be used for the functional \mathcal{L} in (1.2). Actually, since Morse [40] one has been applying the Morse theory, by his classical method of finite dimensional approximations, to studies of geodesics for Finsler metrics [2, 35, 42, 44]. For example, the use of the shifting theorem of Gromoll-Meyer [24] was carried out with it in the generalization of a famous theorem of Gromoll-Meyer [25] to Finsler manifolds in [35]. However, the finite dimensional approximation of the loop space $\Lambda M = \Lambda_{\Delta_M}(M)$ by spaces of broken geodesics carries only a \mathbb{Z}_k but not a S^1 action. It is hard to apply this classical method to the studies of some geodesic problems.

Notation. For a normed vector space $(E, \|\cdot\|)$ and $\delta > 0$ we write $\mathbf{B}_\delta(E) = \{x \in E \mid \|x\| < \delta\}$ and $\bar{\mathbf{B}}_\delta(E) = \{x \in E \mid \|x\| \leq \delta\}$ (since the notations of some spaces involved are complex). For a continuous symmetric bilinear form (or the associated self-adjoint operator) B on a Hilbert space we write $\mathbf{H}^-(B)$, $\mathbf{H}^0(B)$ and $\mathbf{H}^+(B)$ for the negative definite, null and positive definite spaces of it. \mathbb{K} always denotes an Abelian group without special statements.

In this paper we only consider the following two cases:

- $N = M_0 \times M_1$, where M_0 and M_1 are two disjoint boundaryless submanifolds of M . In this case the boundary condition in Proposition 1.1(ii) becomes

$$\begin{cases} g^F(\gamma(0), \dot{\gamma}(0))[u, \dot{\gamma}(0)] = 0 & \forall u \in T_{\gamma(0)}M_0, \\ g^F(\gamma(1), \dot{\gamma}(1))[v, \dot{\gamma}(1)] = 0 & \forall v \in T_{\gamma(1)}M_1. \end{cases} \quad (1.3)$$

- $N = \Delta_M$ and hence $\Lambda_N(M) = W^{1,2}(S^1, M)$, where $S^1 = \mathbb{R}/\mathbb{Z} = [0, 1]/\{0, 1\}$.

1.1 The case $N = M_0 \times M_1$

Suppose that $\gamma_0 \in \Lambda_N(M)$ is an isolated nonconstant critical point of \mathcal{L} on $\Lambda_N(M)$. By Proposition 1.1(ii) γ_0 is a C^k -smooth nonconstant F -geodesics with constant speed $F(\gamma_0(t), \dot{\gamma}_0(t)) \equiv \sqrt{c} > 0$. Clearly, there exists an open neighborhood of γ_0 in $\Lambda_N(M)$, $\mathcal{O}(\gamma_0)$, and a compact neighborhood K of $\gamma_0(I)$ in M such that each curve of $\mathcal{O}(\gamma_0)$ has an image set contained in K . Thus we shall assume M to be compact below. Moreover, we can require the Riemannian metric h on M to have the property that M_0 (resp. M_1) is totally geodesic near $\gamma_0(0)$ (resp. $\gamma_0(1)$). Let \exp denote the exponential map of h , and let $\mathbf{B}_{2\rho}(T_{\gamma_0}\Lambda_N(M)) = \{\xi \in T_{\gamma_0}\Lambda_N(M) \mid \|\xi\|_{W^{1,2}} < 2\rho\}$ for $\rho > 0$. If ρ is small enough we get a coordinate chart around γ_0 on $\Lambda_N(M)$:

$$\text{EXP}_{\gamma_0} : \mathbf{B}_{2\rho}(T_{\gamma_0}\Lambda_N(M)) \rightarrow \Lambda_N(M) \quad (1.4)$$

given by $\text{EXP}_{\gamma_0}(\xi)(t) = \exp_{\gamma_0(t)}(\xi(t))$. Then $\mathcal{L} \circ \text{EXP}_{\gamma_0}$ is C^{2-0} and has an isolated critical point $0 \in W_N^{1,2}(\gamma_0^*TM)$. Consider the Banach manifold

$$\mathcal{X} = C_N^1(I, M) = \{\gamma \in C^1(I, M) \mid (\gamma(0), \gamma(1)) \in N\}.$$

Then the tangent space $T_{\gamma_0}\mathcal{X} = C_{TN}^1(\gamma_0^*TM) = \{\xi \in C^1(\gamma_0^*TM) \mid (\xi(0), \xi(1)) \in TN\}$. Let \mathcal{A} be the restriction of the gradient of $\mathcal{L} \circ \text{EXP}_{\gamma_0}$ to $\mathbf{B}_{2\rho}(T_{\gamma_0}\Lambda_N(M)) \cap T_{\gamma_0}\mathcal{X}$. Obverse that $\mathbf{B}_{2\rho}(T_{\gamma_0}\Lambda_N(M)) \cap T_{\gamma_0}\mathcal{X}$ is an open neighborhood of 0 in $T_{\gamma_0}\mathcal{X}$ and that \mathcal{A} is a C^1 -map from a small neighborhood of 0 in $T_{\gamma_0}\mathcal{X}$ to $T_{\gamma_0}\mathcal{X}$ by (4.28) and (4.19). Moreover, $\mathcal{A}(0) = \nabla \mathcal{L}(\gamma_0)|_{T_{\gamma_0}\mathcal{X}}$ and

$$\langle D\mathcal{A}(0)\xi, \eta \rangle_1 = d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)(\xi, \eta) \quad \forall \xi, \eta \in C_{TN}^1(\gamma_0^*TM).$$

The key is that the symmetric bilinear form $d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)$ can be extended into such a form on $T_{\gamma_0}\Lambda_N(M)$, also denoted by $d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)$. The self-adjoint operator associated to the latter is Fredholm, and has finite dimensional negative definite and null spaces $\mathbf{H}^-(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0))$ and $\mathbf{H}^0(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0))$ by (4.29), which are actually contained in $C_{TN}^1(\gamma_0^*TM)$. There exists an orthogonal decomposition

$$T_{\gamma_0}\Lambda_N(M) = \mathbf{H}^-(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)) \oplus \mathbf{H}^0(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)) \oplus \mathbf{H}^+(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)) \quad (1.5)$$

according to the negative, null and positive definiteness of $d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)$. It induces a (topological) direct sum decomposition of Banach spaces

$$C_{TN}^1(\gamma_0^*TM) = \mathbf{H}^-(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)) \dot{+} \mathbf{H}^0(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)) \dot{+} (\mathbf{H}^+(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)) \cap C_{TN}^1(\gamma_0^*TM)).$$

Using the implicit function theorem we get $\delta \in (0, 2\rho]$ and a unique C^1 -map

$$h : \mathbf{B}_\delta(\mathbf{H}^0(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0))) \rightarrow \mathbf{H}^-(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)) \dot{+} (\mathbf{H}^+(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)) \cap C_{TN}^1(\gamma_0^*TM))$$

such that $h(0) = 0$ and $(I - P^0)\mathcal{A}(\xi + h(\xi)) = 0$ for any $\xi \in \mathbf{B}_\delta(\mathbf{H}^0(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0))) = \{\xi \in \mathbf{H}^0(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)) \mid \|\xi\|_{W^{1,2}} < \delta\}$, where $P^* : T_{\gamma_0}\Lambda_N(M) \rightarrow \mathbf{H}^*(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0))$, $\star = -, 0, +$, are the orthogonal projections according to (1.5). Define

$$\mathcal{L}^\circ(\xi) = \mathcal{L} \circ \text{EXP}_{\gamma_0}(\xi + h(\xi)) = \mathcal{L}(\exp_{\gamma_0}(\xi + h(\xi))) \quad (1.6)$$

for $\xi \in \mathbf{B}_\delta(\mathbf{H}^0(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)))$ (shrinking $\delta > 0$ if necessary). It is C^2 and has an isolated critical point 0 by (4.22) and (4.31). Denote by $C_*(\mathcal{L}, \gamma_0; \mathbb{K})$ (resp. $C_*(\mathcal{L}^\circ, 0; \mathbb{K})$) the critical group of the functional \mathcal{L} (resp. \mathcal{L}°) at γ_0 (resp. 0) with coefficient group \mathbb{K} . Note that

$$m^-(\gamma_0) := \dim \mathbf{H}^-(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)) \quad \text{and} \quad m^0(\gamma_0) := \dim \mathbf{H}^0(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0))$$

do not depend on the choice of h , called **Morse index** and **nullity** of γ_0 , respectively. Here is our first key result.

Theorem 1.2 *Under the above notations, there exists a C^k convex quadratic growth Lagrangian $L^* : TM \rightarrow \mathbb{R}$ such that:*

- (i) $L^* \leq L$, $L^*(x, v) = L(x, v)$ if $L(x, v) \geq 2c/3$, and if F is reversible so is L^* .
- (ii) The corresponding functional \mathcal{L}^* in (1.24) is C^{2-0} . All functional $\mathcal{L}^\tau = (1 - \tau)\mathcal{L} + \tau\mathcal{L}^*$, $\tau \in [0, 1]$, have only a critical point γ_0 in some neighborhood of $\gamma_0 \in \Lambda_N(M)$, and satisfy the Palais-Smale condition. Moreover, $C_*(\mathcal{L}^*, \gamma_0; \mathbb{K}) = C_*(\mathcal{L}, \gamma_0; \mathbb{K})$.
- (iii) By shrinking the above $\delta > 0$ there exists an origin-preserving homeomorphism ψ from $\mathbf{B}_\delta(T_{\gamma_0}\Lambda_N(M))$ to an open neighborhood of 0 in $T_{\gamma_0}\Lambda_N(M)$ such that

$$\mathcal{L}^* \circ \text{EXP}_{\gamma_0} \circ \psi(\xi) = \langle P^+\xi, P^+\xi \rangle_1 - \langle P^-\xi, P^-\xi \rangle_1 + \mathcal{L}^\circ(P^0\xi)$$

for all $\xi \in \mathbf{B}_\delta(T_{\gamma_0}\Lambda_N(M))$. Moreover, $\psi((P^- + P^0)\mathbf{B}_\delta(T_{\gamma_0}\Lambda_N(M)))$ is contained in $T_{\gamma_0}\mathcal{X}$, and ψ is also a homeomorphism from $(P^- + P^0)\mathbf{B}_\delta(T_{\gamma_0}\Lambda_N(M))$ onto $\psi((P^- + P^0)\mathbf{B}_\delta(T_{\gamma_0}\Lambda_N(M)))$ even if the topology on the latter is taken as the induced one by $T_{\gamma_0}\mathcal{X}$.

- (iv) For any open neighborhood W of γ_0 in $\Lambda_N(M)$ and a field \mathbb{K} , write $W_X := W \cap \mathcal{X}$ as an open subset of \mathcal{X} , then the inclusion

$$((\mathcal{L}^*|_{\mathcal{X}})_c \cap W_X, (\mathcal{L}^*|_{\mathcal{X}})_c \cap W_X \setminus \{0\}) \hookrightarrow (\mathcal{L}_c^* \cap W, \mathcal{L}_c^* \cap W \setminus \{0\})$$

induces isomorphisms

$$H_*((\mathcal{L}^*|_{\mathcal{X}})_c \cap W_X, (\mathcal{L}^*|_{\mathcal{X}})_c \cap W_X \setminus \{0\}; \mathbb{K}) \rightarrow H_*(\mathcal{L}_c^* \cap W, \mathcal{L}_c^* \cap W \setminus \{0\}; \mathbb{K}).$$

For $\mathcal{L}|_{\mathcal{X}}$ we have the splitting lemma at γ_0 as follows.

Theorem 1.3 *Under the above notations, there exist $\epsilon \in (0, \delta)$ and an origin-preserving homeomorphism φ from $\mathbf{B}_\epsilon(C_{TN}^1(\gamma_0^*TM))$ to an open neighborhood of 0 in $C_{TN}^1(\gamma_0^*TM)$ such that*

$$\mathcal{L} \circ \text{EXP}_{\gamma_0} \circ \varphi(\xi) = \frac{1}{2} d^2 \mathcal{L}|_{\mathcal{X}}(\gamma_0)(P^+ \xi, P^+ \xi) - \|P^- \xi\|_1^2 + \mathcal{L}^\circ(\xi^0)$$

for all $\xi \in \mathbf{B}_\epsilon(C_{TN}^1(\gamma_0^*TM))$, where ξ^\star , $\star = -, 0, +$, are the projections of ξ onto $\mathbf{H}^\star(d^2 \mathcal{L}|_{\mathcal{X}}(\gamma_0))$ according to the orthogonal decomposition (1.5).

By Corollaries A.2, A.6 Theorem 1.2(iii) and Theorem 1.3 give two results of the following shifting theorem, respectively.

Theorem 1.4 *For all $q = 0, 1, \dots$ it holds that*

$$\begin{aligned} C_q(\mathcal{L}, \gamma_0; \mathbb{K}) &= C_{q-m^-(\gamma_0)}(\mathcal{L}^\circ, 0; \mathbb{K}) \quad \text{and} \\ C_q(\mathcal{L}|_{\mathcal{X}}, \gamma_0; \mathbb{K}) &= C_{q-m^-(\gamma_0)}(\mathcal{L}^\circ, 0; \mathbb{K}). \end{aligned} \quad (1.7)$$

As a consequence of this result we have

$$C_*(\mathcal{L}|_{\mathcal{X}}, \gamma_0; \mathbb{K}) = C_*(\mathcal{L}, \gamma_0; \mathbb{K}). \quad (1.8)$$

Note that Theorem 1.2(iv) is stronger than this. Indeed, if we choose a small open neighborhood V of γ_0 in \mathcal{X} such that the closure of V in $\Lambda_N(M)$ (resp. \mathcal{X}) is contained in W (resp. the open subset W_X of \mathcal{X}) and that $\min_t L(\gamma(t), \dot{\gamma}(t)) \geq 2c/3$ (and hence $\mathcal{L}^*(\gamma) = \mathcal{L}(\gamma)$) for any $\gamma \in V$, then

$$H_*((\mathcal{L}^*|_{\mathcal{X}})_c \cap W_X, (\mathcal{L}^*|_{\mathcal{X}})_c \cap W_X \setminus \{0\}; \mathbb{K}) = H_*((\mathcal{L}|_{\mathcal{X}})_c \cap V, (\mathcal{L}|_{\mathcal{X}})_c \cap V \setminus \{0\}; \mathbb{K}).$$

by the excision property of the relative homology groups. This and Theorem 1.2(ii),(iv) imply (1.8). In applications it is more effective combining these results together.

When M_0 and M_1 are two disjoint points, Caponio-Javaloyes-Masiello [11, Th.7] proved a splitting lemma for $\mathcal{L}|_{\mathcal{X}}$ around γ_0 as Theorem 1.3 (in some possible different coordinate chart) and thus obtained the shifting theorem (1.7) with a different method, and then used Chang's ideas of [14] to prove (1.8) in [12] by Palais' theorems 16 and 17 in [41]. See Remark 4.6 for a detailed illustration.

1.2 The case $N = \triangle_M$

Now $\Lambda_N(M) = \Lambda M := W^{1,2}(S^1, M)$. Hereafter $S^1 := \mathbb{R}/\mathbb{Z} = \{[s] \mid [s] = s + \mathbb{Z}, s \in \mathbb{R}\}$. There exist equivariant and also isometric operations of S^1 -action on $W^{1,2}(S^1, M)$ and $TW^{1,2}(S^1, M)$:

$$[s] \cdot \gamma(t) = \gamma(s + t), \quad \forall [s] \in S^1, \gamma \in W^{1,2}(S^1, M), \quad (1.9)$$

$$[s] \cdot \xi(t) = \xi(s + t), \quad \forall [s] \in S^1, \xi \in T_\gamma W^{1,2}(S^1, M), \quad (1.10)$$

which are continuous, but not differentiable ([27, Chp.2, §2.2]). Since \mathcal{L} is invariant under the S^1 -action, a nonconstant curve $\gamma \in \Lambda M$ cannot be an isolated critical point

of the functional \mathcal{L} . Let $\gamma_0 \in \Lambda M$ be a (nonconstant) critical point of \mathcal{L} with critical value $c > 0$. Under our assumptions γ_0 is C^k -smooth by Proposition 3.1. The orbit $S^1 \cdot \gamma_0$ is a C^k -submanifold in $\Lambda M = W^{1,2}(S^1, M)$ by [25, page 499], and hence a C^k -smooth critical submanifold of \mathcal{L} in ΛM . We assume that $S^1 \cdot \gamma_0$ is an isolated critical orbit and $k \geq 4$ below.

Also write $\mathcal{X} = C^1(S^1, M)$ in this subsection. Let $\mathcal{O} = S^1 \cdot \gamma_0$ and let $\pi : N\mathcal{O} \rightarrow \mathcal{O}$ be the normal bundle of it in ΛM . It is a Hilbert vector bundle over \mathcal{O} , and $XN\mathcal{O} := T_{\mathcal{O}}\mathcal{X} \cap N\mathcal{O}$ is a Banach vector bundle over \mathcal{O} . For $\varepsilon > 0$ we denote by

$$\left. \begin{aligned} N\mathcal{O}(\varepsilon) &= \{(x, v) \in N\mathcal{O} \mid \|v\|_{W^{1,2}} < \varepsilon\}, \\ XN\mathcal{O}(\varepsilon) &= \{(x, v) \in XN\mathcal{O} \mid \|v\|_{C^1} < \varepsilon\}. \end{aligned} \right\} \quad (1.11)$$

Clearly, $XN\mathcal{O}(\varepsilon) \subset N\mathcal{O}(\varepsilon)$. For sufficiently small $\varepsilon > 0$, the exponential map \exp gives a map

$$\text{EXP} : T\Lambda M(\varepsilon) = \{(x, v) \in T\Lambda M \mid \|v\|_{W^{1,2}} < \varepsilon\} \rightarrow \Lambda M \quad (1.12)$$

defined by $\text{EXP}(x, v)(t) = \exp_{x(t)} v(t) \forall t \in \mathbb{R}$, which restricts to a diffeomorphism from the normal disk bundle $N\mathcal{O}(\varepsilon)$ onto an open neighborhood of \mathcal{O} in ΛM , $\mathcal{N}(\mathcal{O}, \varepsilon)$. For $x \in \mathcal{O}$ let $N\mathcal{O}(\varepsilon)_x$ and $XN\mathcal{O}(\varepsilon)_x$ be the fibers of $N\mathcal{O}(\varepsilon)$ and $XN\mathcal{O}(\varepsilon)$ at $x \in \mathcal{O}$, respectively. Denote by A_x the restriction of the gradient $\nabla(\mathcal{L} \circ \text{EXP}|_{N\mathcal{O}(\varepsilon)_x})$ to $N\mathcal{O}(\varepsilon)_x \cap XN\mathcal{O}_x$. Then for $\delta > 0$ small enough A_x is C^1 in $XN\mathcal{O}(\delta)_x$ (and so $\mathcal{L} \circ \text{EXP}|_{XN\mathcal{O}(\delta)_x}$ is C^2) by (5.12), (5.39), (5.42) and (B.18). Moreover it is clear that

$$A_{s \cdot x}(s \cdot v) = s \cdot A_x(v) \quad \forall s \in S^1, v \in N\mathcal{O}(\varepsilon)_x \cap XN\mathcal{O}_x. \quad (1.13)$$

Denote by B_x the symmetric bilinear form $d^2(\mathcal{L} \circ \text{EXP}|_{XN\mathcal{O}(\varepsilon)_x})(0_x)$. By (i) above Claim 5.9 we see that it can be extended into such a form on $N\mathcal{O}_x$, also denoted by B_x , whose associated self-adjoint operator is Fredholm, and has finite dimensional negative definite and null spaces $\mathbf{H}^-(B_x)$ and $\mathbf{H}^0(B_x)$. Moreover, $\mathbf{H}^-(B_x) + \mathbf{H}^0(B_x)$ is contained in $XN\mathcal{O}_x$. As before there exists an orthogonal decomposition

$$N\mathcal{O}_x = \mathbf{H}^-(B_x) \oplus \mathbf{H}^0(B_x) \oplus \mathbf{H}^+(B_x) \quad (1.14)$$

(according to the negative definite, null and positive of B_x). Since $B_{s \cdot x}(s \cdot \xi, s \cdot \eta) = B_x(\xi, \eta)$ for any $s \in S^1$ and $x \in \mathcal{O}$, (1.14) leads to a natural C^2 Hilbert vector bundle orthogonal decomposition

$$N\mathcal{O} = \mathbf{H}^-(B) \oplus \mathbf{H}^0(B) \oplus \mathbf{H}^+(B) \quad (1.15)$$

with $\mathbf{H}^*(B)_x = \mathbf{H}^*(B_x)$ for $x \in \mathcal{O}$ and $\star = +, 0, -$, which induces a C^2 Banach vector bundle (topological) direct sum decomposition

$$XN\mathcal{O} = \mathbf{H}^-(B) \dot{+} \mathbf{H}^0(B) \dot{+} (\mathbf{H}^+(B) \cap XN\mathcal{O}). \quad (1.16)$$

We call $m^-(\mathcal{O}) := \text{rank} \mathbf{H}^-(B)$ and $m^0(\mathcal{O}) := \text{rank} \mathbf{H}^0(B)$ **Morse index** and **nullity** of $\mathcal{O} = S^1 \cdot \gamma_0$, respectively. In the case $m^0(\mathcal{O}) = 0$ the orbit \mathcal{O} is said to

be **nondegenerate**. Moreover we have always $0 \leq m^0(\mathcal{O}) \leq 2n - 1$. (Indeed, $m^0(\mathcal{O}) = \text{rank} \mathbf{H}^0(\hat{B}^*) = m^0(\gamma_0) - 1 \leq 2n - 1$ by (5.8), (5.13) and the inequality above Theorem 3.1 of [31].)

Let \mathbf{P}^\star be the orthogonal bundle projections from $N\mathcal{O}$ onto $\mathbf{H}^\star(B)$, $\star = +, 0, -$, and let $\mathbf{H}^0(B)(\epsilon) = \mathbf{H}^0(B) \cap N\mathcal{O}(\epsilon)$ for $\epsilon > 0$. Note that $\mathbf{H}^0(B)(\epsilon) \subset XN\mathcal{O}$ and that for any $\delta > 0$ we may choose $\epsilon > 0$ so small that $\mathbf{H}^0(B)(\epsilon) \subset XN\mathcal{O}(\delta)$ since $\mathbf{H}^0(B)$ has finite rank. By the implicit function theorem and the compactness of \mathcal{O} , if $\epsilon > 0$ is sufficiently small for each $x \in \mathcal{O}$ there exists a unique S_x^1 -equivariant C^1 map

$$\mathfrak{h}_x : \mathbf{H}^0(B)(\epsilon)_x \rightarrow \mathbf{H}^-(B)_x + (\mathbf{H}^+(B)_x \cap XN\mathcal{O}_x) \quad (1.17)$$

satisfying $\mathfrak{h}_x(0_x) = 0_x$ and

$$(\mathbf{P}_x^+ + \mathbf{P}_x^-) \circ A_x(v + \mathfrak{h}_x(v)) = 0 \quad \forall v \in \mathbf{H}^0(B)(\epsilon)_x. \quad (1.18)$$

Moreover, by (B.14) and (5.39) the functional

$$\mathcal{L}_\Delta^\circ : \mathbf{H}^0(B)(\epsilon) \ni (x, v) \rightarrow \mathcal{L} \circ \text{EXP}_x(v + \mathfrak{h}_x(v)) \in \mathbb{R} \quad (1.19)$$

restricts to a C^2 one in each fiber $\mathbf{H}^0(B)(\epsilon)_x$, denote by $\mathcal{L}_{\Delta x}^\circ$ for $x \in \mathcal{O}$, and it has the isolated critical orbit \mathcal{O} .

Theorem 1.5 *Under the above notations, there exists a C^k convex quadratic growth Lagrangian $L^* : TM \rightarrow \mathbb{R}$ such that:*

- (i) $L^* \leq L$, $L^*(x, v) = L(x, v)$ if $L(x, v) \geq 2c/3$, and if F is reversible so is L^* .
- (ii) The corresponding functional \mathcal{L}^* in (1.24) is C^{2-0} in ΛM . All functional $\mathcal{L}^\tau = (1 - \tau)\mathcal{L} + \tau\mathcal{L}^*$, $\tau \in [0, 1]$, have only a critical orbit \mathcal{O} in some neighborhood of $\mathcal{O} \subset \Lambda M$, and satisfy the Palais-Smale condition. Moreover, $C_*(\mathcal{L}^*, \mathcal{O}; \mathbb{K}) = C_*(\mathcal{L}, \mathcal{O}; \mathbb{K})$.
- (iii) By shrinking the above $\epsilon > 0$ (if necessary) there exist a S^1 -invariant open neighborhood U of the zero section of $N\mathcal{O}$, a S^1 -equivariant fiber-preserving, continuous and fibrewise C^1 map \mathfrak{h} given by (1.17) and (1.18), and a S^1 -equivariant fiber-preserving homeomorphism $\Upsilon : N\mathcal{O}(\epsilon) \rightarrow U$ such that

$$\mathcal{L}^* \circ \text{EXP} \circ \Upsilon(u) = \|\mathbf{P}^+ u\|_1^2 - \|\mathbf{P}^- u\|_1^2 + \mathcal{L}_\Delta^\circ(\mathbf{P}^0 u)$$

for all $u \in N\mathcal{O}(\epsilon)$. Moreover, $\Upsilon((\mathbf{P}^- + \mathbf{P}^0)N\mathcal{O}(\epsilon))$ is contained in $XN\mathcal{O}$, and Υ is also a homeomorphism from $(\mathbf{P}^- + \mathbf{P}^0)N\mathcal{O}(\epsilon)$ onto $\Upsilon((\mathbf{P}^- + \mathbf{P}^0)N\mathcal{O}(\epsilon))$ even if the topology on the latter is taken as the induced one by $XN\mathcal{O}$. (This implies that $N\mathcal{O}$ and $XN\mathcal{O}$ induce the same topology in $\Upsilon((\mathbf{P}^- + \mathbf{P}^0)N\mathcal{O}(\epsilon))$.)

- (iv) For any open neighborhood W of \mathcal{O} in ΛM and a field \mathbb{K} , write $W_X = W \cap \mathcal{X}$ as an open subset of \mathcal{X} , then the inclusion

$$((\mathcal{L}^*|_{\mathcal{X}})_c \cap W_X, (\mathcal{L}^*|_{\mathcal{X}})_c \cap W_X \setminus \mathcal{O}) \hookrightarrow (\mathcal{L}_c^* \cap W, \mathcal{L}_c^* \cap W \setminus \mathcal{O})$$

induces isomorphisms

$$H_*((\mathcal{L}^*|_{\mathcal{X}})_c \cap W_X, (\mathcal{L}^*|_{\mathcal{X}})_c \cap W_X \setminus \mathcal{O}; \mathbb{K}) \rightarrow H_*(\mathcal{L}_c^* \cap W, \mathcal{L}_c^* \cap W \setminus \mathcal{O}; \mathbb{K}).$$

Moreover, the corresponding conclusion can be obtained if

$$((\mathcal{L}^*|_{\mathcal{X}})_c \cap W_X, (\mathcal{L}^*|_{\mathcal{X}})_c \cap W_X \setminus \mathcal{O}) \quad \text{and} \quad (\mathcal{L}_c^* \cap W, \mathcal{L}_c^* \cap W \setminus \mathcal{O})$$

are replaced by $((\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c \cap W_X \cup \mathcal{O}, (\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c \cap W_X)$ and $(\mathring{\mathcal{L}}_c^* \cap W \cup \mathcal{O}, \mathring{\mathcal{L}}_c^* \cap W)$, respectively, where $\mathring{\mathcal{L}}_c^* = \{\mathcal{L}^* < c\}$ and $(\mathring{\mathcal{L}}^*|_{\mathcal{X}})_c = \{\mathcal{L}^*|_{\mathcal{X}} < c\}$.

(v) If $\mathcal{L}_{\mathcal{X}}^*$ and \mathcal{L}^* in (iv) are replaced by $\mathcal{L}_{\mathcal{X}}$ and \mathcal{L} , respectively, then the corresponding conclusion also holds true.

Corresponding to Theorem 1.3 we have the following splitting lemma for $\mathcal{L}|_{\mathcal{X}}$ around \mathcal{O} as follows.

Theorem 1.6 *Under the notations above, by shrinking the above $\epsilon > 0$ there exist a S^1 -invariant open neighborhood V of the zero section of $XN\mathcal{O}$, a S^1 -equivariant fiber-preserving, continuous and fibrewise C^1 map \mathfrak{h} given by (1.17) and (1.18), and a S^1 -equivariant fiber-preserving homeomorphism $\Psi : XN\mathcal{O}(\epsilon) \rightarrow V$ such that*

$$\mathcal{L} \circ \text{EXP} \circ \Psi(x, v) = \frac{1}{2} d^2 \mathcal{L}|_{\mathcal{X}}(x)(\mathbf{P}_x^+ v, \mathbf{P}_x^+ v) - \|\mathbf{P}_x^- v\|_1^2 + \mathcal{L}_{\Delta x}^{\circ}(\mathbf{P}_x^0 v)$$

for all $(x, v) \in XN\mathcal{O}(\epsilon)$.

Clearly, Theorem 1.5(v) implies

$$C_*(\mathcal{L}, \mathcal{O}; \mathbb{K}) = C_*(\mathcal{L}|_{\mathcal{X}}, \mathcal{O}; \mathbb{K}). \quad (1.20)$$

This also easily follows from Theorem 1.5(i),(ii) and (iv). Let $\mathbf{H}^{0-}(B) = \mathbf{H}^0(B) + \mathbf{H}^-(B)$ and $\mathbf{H}^{0-}(B)(\epsilon) = (\mathbf{H}^0(B) + \mathbf{H}^-(B)) \cap N\mathcal{O}(\epsilon)$. Then $\mathbf{H}^{0-}(B) \subset XN\mathcal{O}$. Define $\mathfrak{L} : \mathbf{H}^{0-}(B)(\epsilon) \rightarrow \mathbb{R}$ by

$$\mathfrak{L}(x, v) = -\|\mathbf{P}_x^- v\|_1^2 + \mathcal{L}_{\Delta x}^{\circ}(\mathbf{P}_x^0 v).$$

By the obvious deformation arguments we may derive from Theorem 1.5(ii)-(iii) and Theorem 1.6 respectively:

$$C_*(\mathcal{L}, \mathcal{O}; \mathbb{K}) = C_*(\mathfrak{L}, \mathcal{O}; \mathbb{K}) \quad \text{and} \quad C_*(\mathcal{L}|_{\mathcal{X}}, \mathcal{O}; \mathbb{K}) = C_*(\mathfrak{L}, \mathcal{O}; \mathbb{K}).$$

These give rise to (1.20) again.

Let $S_x^1 \subset S^1$ denote the stabilizer of $x \in \mathcal{O}$. Since x is nonconstant S_x^1 is a finite cyclic group and the quotient $S^1/S_x^1 \cong S^1 \cdot \gamma_0 = \mathcal{O} \cong S^1$. (See [25, page 499]). Clearly, $\mathcal{L}_{\Delta x}^{\circ}$ is S_x^1 -invariant. Let $C_*(\mathcal{L}_{\Delta x}^{\circ}, 0; \mathbb{K})^{S_x^1}$ denote the subgroup of all elements in $C_*(\mathcal{L}_{\Delta x}^{\circ}, 0; \mathbb{K})$, which are fixed by the induced action of S_x^1 on the homology. We have the following generalization of the Gromoll-Meyer shifting theorem for Finsler manifolds.

Theorem 1.7 *Let \mathbb{K} be a field of characteristic 0 or prime to order $|S_{\gamma_0}^1|$ of $S_{\gamma_0}^1$. Then for any $x \in \mathcal{O}$ and $q = 0, 1, \dots$,*

$$\begin{aligned} & C_q(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) \\ &= \left(H_{m^-(S^1 \cdot \gamma_0)}(\mathbf{H}^-(B)_x, \mathbf{H}^-(B)_x \setminus \{0_x\}; \mathbb{K}) \otimes C_{q-m^-(S^1 \cdot \gamma_0)-1}(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}) \right)^{S_x^1} \\ &\oplus \left(H_{m^-(S^1 \cdot \gamma_0)}(\mathbf{H}^-(B)_x, \mathbf{H}^-(B)_x \setminus \{0_x\}; \mathbb{K}) \otimes C_{q-m^-(S^1 \cdot \gamma_0)}(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}) \right)^{S_x^1}. \end{aligned}$$

provided $m^-(S^1 \cdot \gamma_0)m^0(S^1 \cdot \gamma_0) > 0$. Moreover,

$$C_q(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) = (C_{q-1}(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1} \oplus (C_q(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1}$$

if $m^-(S^1 \cdot \gamma_0) = 0$ and $m^0(S^1 \cdot \gamma_0) > 0$, and

$$\begin{aligned} C_q(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) &= H_q(\mathbf{H}^-(B), \mathbf{H}^-(B) \setminus \mathcal{O}; \mathbb{K}) \\ &= \left(H_{q-1}(\mathbf{H}^-(B)_x, \mathbf{H}^-(B)_x \setminus \{0_x\}; \mathbb{K}) \right)^{S_x^1} \\ &\oplus \left(H_q(\mathbf{H}^-(B)_x, \mathbf{H}^-(B)_x \setminus \{0_x\}; \mathbb{K}) \right)^{S_x^1} \end{aligned}$$

if $m^-(S^1 \cdot \gamma_0) > 0$ and $m^0(S^1 \cdot \gamma_0) = 0$. Finally, $C_q(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) = H_q(S^1; \mathbb{K})$ for any Abelian group \mathbb{K} if $m^-(S^1 \cdot \gamma_0) = m^0(S^1 \cdot \gamma_0) = 0$.

By [6, Th.I] $H_q(\mathbf{H}^-(B), \mathbf{H}^-(B) \setminus \mathcal{O}; \mathbb{Z}_2) = \mathbb{Z}_2$ for $q = m^-(\mathcal{O}), m^-(\mathcal{O}) + 1$, and $= 0$ otherwise. If $\mathbb{K} = \mathbb{Z}$ and $\mathbf{H}^-(B)$ is orientable the same is also true; see [27, Cor.2.4.11].

In the studies of closed geodesics one often define the critical group

$$C_*(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) = H_*(\Lambda(\gamma_0) \cup S^1 \cdot \gamma_0, \Lambda(\gamma_0); \mathbb{K}),$$

where $\Lambda(\gamma_0) = \{\gamma \in \Lambda M \mid \mathcal{L}(\gamma) < \mathcal{L}(\gamma_0)\}$. Using the excision property of singular homology and anti-gradient flow it is not hard to prove that these two kinds of definitions agree (cf. [20, Propositions 3.4 and 3.7]). Such a version of Theorem 1.7 was proved in [2, Prop.3.7] by introducing finite dimensional approximations to ΛM as in [35, 42, 44].

Since one does not know if a generator of the S_x^1 -action on $\mathbf{H}^-(B)_x$ reverses orientation or not, no further explicit version of the formula in Theorem 1.7 can be obtained though

$$H_{m^-(\mathcal{O})}(\mathbf{H}^-(B)_x, \mathbf{H}^-(B)_x \setminus \{0_x\}; \mathbb{K}) = \mathbb{K}.$$

Recalling $S^1 = \mathbb{R}/\mathbb{Z} = [0, 1]/\{0, 1\}$, there exists a positive integer $m = m(\gamma_0)$ such that $1/m$ is the minimal period of γ_0 (since γ_0 is nonconstant). It is equal to the order of the isotropy group $S_{\gamma_0}^1$, and is called the **multiplicity** of γ_0 . When $m(\gamma_0) = 1$ we say γ_0 to be **prime**. These can also be described by the m -th iterate operation

$$\varphi_m : \Lambda M \rightarrow \Lambda M, \gamma \rightarrow \gamma^m \tag{1.21}$$

defined by¹ $\gamma^m(t) = \gamma(mt) \forall t \in \mathbb{R}$ when γ is viewed as a 1-periodic map $\gamma : \mathbb{R} \rightarrow M$. Clearly, there exists a unique prime curve $\gamma_0^{1/m} \in \Lambda M$ such that $(\gamma_0^{1/m})^m$ is equal to γ_0 . Suppose that \mathbb{K} is a field \mathbb{Q} of rational numbers and that for each $k \in \mathbb{N}$ the orbit $S^1 \cdot (\gamma_0^{1/m})^k$ is an isolated critical orbit of \mathcal{L} . We may rewrite the conclusions of Proposition 3.8 in [2] (in our notations) as follows:

- (i) If $m^0(S^1 \cdot \gamma_0) = 0$ then $C_q(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) = \mathbb{K}$ if $m^-(S^1 \cdot \gamma_0) - m^-(S^1 \cdot \gamma_0^{1/m})$ is even and $q \in \{m^-(S^1 \cdot \gamma_0), m^-(S^1 \cdot \gamma_0) + 1\}$, and $C_q(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) = 0$ in other cases.
- (ii) If $m^0(S^1 \cdot \gamma_0) > 0$ and $\epsilon(\gamma_0) = (-1)^{m^-(S^1 \cdot \gamma_0) - m^-(S^1 \cdot \gamma_0^{1/m})}$, then

$$\begin{aligned} C_q(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) &= C_{q-m^-(S^1 \cdot \gamma_0)-1}(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K})^{S_x^1, \epsilon(\gamma_0)} \\ &\oplus C_{q-m^-(S^1 \cdot \gamma_0)}(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K})^{S_x^1, \epsilon(\gamma_0)} \end{aligned}$$

for each $x \in \mathcal{O}$ and $q \in \mathbb{N} \cup \{0\}$, where $C_*(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K})^{S_x^1, 1}$ and $C_*(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K})^{S_x^1, -1}$ are the eigenspaces of a generator of S_x^1 corresponding to 1 and -1 , respectively. Clearly, $C_*(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K})^{S_x^1, -1} = 0$ if m is odd.

Using Theorem 1.7 we may derive the following result, which is very important for the proof of Theorem 1.11 in Section 8.

Theorem 1.8 *Under the assumptions of Theorem 1.7 suppose that $m^-(\gamma_0) = 0$ and that $C_p(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) \neq 0$ and $C_{p+1}(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) = 0$ for some $p \in \mathbb{N} \cup \{0\}$. Then $p > 0$. Furthermore, we have*

- (i) if $p = 1$ then each point of $S^1 \cdot \gamma_0$ is a local minimum of \mathcal{L} ;
- (ii) if $p \geq 2$ then each point of $S^1 \cdot \gamma_0$ is not a local minimum of $\mathcal{L}|_{\mathcal{X}}$ (and thus \mathcal{L}).

1.3 Two iteration theorems

Our shifting theorems, Theorems 1.4, 1.7, are sufficient for computations of critical groups in most of studies of geodesics on a Finsler manifold. In Riemannian geometry they are direct consequences of the splitting theorem. So far we have not obtained the corresponding splitting theorem for the Finsler energy functional \mathcal{L} on the Hilbert manifold $\Lambda_N(M)$. In the studies of multiplicity of closed geodesics on a Riemannian manifold as in [25, 3] etc, one must use the splitting theorem to deduce a change result of critical groups under the iteration map φ_m . We shall give more general versions of Theorems 1.5, 1.6, 1.7 in Section 5.1 so that the following iteration results corresponding with the case of Riemannian geometry can be proved in Section 6.

Theorem 1.9 *For a closed geodesics γ_0 and some integer $m > 1$, suppose that $\mathcal{O} = S^1 \cdot \gamma_0$ and $\varphi_m(\mathcal{O}) = S^1 \cdot \gamma_0^m$ are two isolated critical orbits of \mathcal{L} in ΛM and that $m^-(\mathcal{O}) = m^-(\varphi_m(\mathcal{O}))$ and $m^0(\mathcal{O}) = m^0(\varphi_m(\mathcal{O}))$. Then φ_m induces isomorphisms*

$$(\varphi_m)_* : H_*(\Lambda(\gamma_0) \cup S^1 \cdot \gamma_0, \Lambda(\gamma_0); \mathbb{K}) \rightarrow H_*(\Lambda(\gamma_0^m) \cup S^1 \cdot \gamma_0^m, \Lambda(\gamma_0^m); \mathbb{K}).$$

¹Here γ^m is different from γ^m appearing in the study of Lagrangian Conley conjecture in [29, 31, 32] though using the same notation.

(Consequently, φ_m induces isomorphisms from $C_*(\mathcal{L}, \mathcal{O}; \mathbb{K})$ to $C_*(\mathcal{L}, \varphi_m(\mathcal{O}); \mathbb{K})$.)

Since \mathcal{L} is C^{2-0} and satisfies the (P.S) condition on the Hilbert manifold $H^1(S^1, M)$ it is not hard to give different equivalent forms of Theorem 1.9, which are more convenient in different applications. See Theorems 6.1, 6.2.

Under the weaker assumption that $m^0(\mathcal{O}) = m^0(\varphi_m(\mathcal{O}))$ some results can also be obtained. When the critical orbit $\mathcal{O} = S^1 \cdot \gamma_0$ between (1.9) and Theorem 1.5 is replaced by $\varphi_m(\mathcal{O}) = S^1 \cdot \gamma_0^m$ the corresponding maps with A , B , \mathfrak{h} and \mathcal{L}_Δ° are denoted by mA , mB , ${}^m\mathfrak{h}$ and ${}^m\mathcal{L}_\Delta^\circ$, respectively.

Theorem 1.10 *Let \mathbb{K} be a field. Suppose that $m^0(\mathcal{O}) = m^0(\varphi_m(\mathcal{O}))$. Then*

$$\dim C_q(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}) = \dim C_q({}^m\mathcal{L}_{\Delta x^m}^\circ, 0; \mathbb{K}) \quad (1.22)$$

for any $x \in \mathcal{O}$ and all $q \in \{0\} \cup \mathbb{N}$. If the characteristic of \mathbb{K} is zero or prime to order of $S_{\gamma_0^m}^1$ then

$$\dim C_q(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K})^{S_x^1} = \dim C_q({}^m\mathcal{L}_{\Delta x^m}^\circ, 0; \mathbb{K})^{S_{x^m}^1} \quad \forall q \in \{0\} \cup \mathbb{N}. \quad (1.23)$$

For the case of closed geodesics on Riemannian manifolds the corresponding result of Theorem 1.9 was contained in the proof of [3, Th.3], (1.22) may be viewed as the corresponding result with Theorem 3 of [25]. With finite-dimensional approximations Theorem 1.9 and the equivalent forms of (1.22)-(1.23) were proved in [42, §7.1, 7.2] and [2, Th.3.11].

1.4 An application

Using the above theory many results about closed geodesics on Riemannian manifolds can be generalized to Finsler manifolds in a straightforward way. For example, repeating the arguments of [3] will lead to similar results. In particular we have the following generalization version of [3, Theorem 3].

Theorem 1.11 *A connected closed Finsler manifold (M, F) of dimension $n > 1$ has infinitely many geometrically distinct closed geodesics provided that there exists a nonconstant closed geodesics $\bar{\gamma}$ such that $m^-(\bar{\gamma}^k) \equiv 0$ and $H_{\bar{p}}(\Lambda(\bar{\gamma}) \cup S^1 \cdot \bar{\gamma}, \Lambda(\bar{\gamma}); \mathbb{K}) \neq 0$ with some integer $\bar{p} \geq 2$ and a field \mathbb{K} of characteristic zero.*

The readers may compare it with Rademacher's generalization [42, Theorem 7.5] with finite-dimensional approximations. Our proof method is slightly different from [3], and cannot deal with the case that $\bar{p} = 1$ and $\bar{\gamma}$ is a local minimum of \mathcal{L} , but not an absolute minimum of \mathcal{L} in its free homotopy class. See Remark 8.7 for comparisons with the results in [3].

1.5 Basic proof idea

Since $W^{1,2}(I, M) \hookrightarrow C^0(I, M)$ is continuous, and each $\gamma \in W^{1,2}(I, M)$ has compact image, there exists an open neighborhood $\mathcal{O}(\gamma)$ of γ in $W^{1,2}(I, M)$ such that $\cup\{\alpha(I) \mid \alpha \in \mathcal{O}(\gamma)\}$ is contained in a compact subset of M . Hence as before we may assume that M is compact without special statements.

Recall that a Lagrangian $L : [0, 1] \times TM \rightarrow \mathbb{R}$ is called **convex quadratic growth** if it satisfies the conditions:

(L1) \exists a constant $\ell_0 > 0$ such that $\partial_{vv}L(t, x, v) \geq \ell_0 I$,

(L2) \exists a constant $\ell_1 > 0$ such that $\|\partial_{vv}L(t, x, v)\| \leq \ell_1$ and

$$\|\partial_{xv}L(t, x, v)\| \leq \ell_1(1 + |v|_x), \quad \|\partial_{xx}L(t, x, v)\| \leq \ell_1(1 + |v|_x^2)$$

with respect to some Riemannian metric $\langle \cdot, \cdot \rangle$ (with $|v|_x^2 = \langle v, v \rangle_x$).

Equivalently, there exists a finite atlas on M such that under each chart of this atlas the following conditions hold for some constants $0 < c < C$:

(L1) $\sum_{ij} \frac{\partial^2}{\partial v_i \partial v_j} L(t, x, v) u_i u_j \geq c |\mathbf{u}|^2 \quad \forall \mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$,

(L2) $\left| \frac{\partial^2}{\partial x_i \partial x_j} L(t, x, v) \right| \leq C(1 + |v|^2), \quad \left| \frac{\partial^2}{\partial x_i \partial v_j} L(t, x, v) \right| \leq C(1 + |v|), \quad \text{and}$
 $\left| \frac{\partial^2}{\partial v_i \partial v_j} L(t, x, v) \right| \leq C.$

To show our ideas let us consider Theorem 1.4 for example. Given a nontrivial constant speed F -geodesic $\gamma_0 : I \rightarrow M$ with $(\gamma_0(0), \gamma_0(1)) \in N = M_0 \times M_1$, $c = L(\gamma_0, \dot{\gamma}_0) = [F(\gamma_0, \dot{\gamma}_0)]^2 > 0$. Then we construct a convex quadratic growth Lagrangian $L^* : TM \rightarrow \mathbb{R}$ such that $L^*(x, v) = L(x, v)$ if $L(x, v) \geq \frac{2c}{3}$. Apparently, γ_0 is a critical point of the functional

$$\mathcal{L}^* : \Lambda_N(M) \rightarrow \mathbb{R}, \quad \gamma \mapsto \int_0^1 L^*(\gamma(t), \dot{\gamma}(t)) dt \quad (1.24)$$

with critical value c . Moreover, γ_0 is isolated for \mathcal{L} if and only if it is so for \mathcal{L}^* . Define $L^\tau(x, v) = (1 - \tau)L(x, v) + \tau L^*(x, v)$ and $\mathcal{L}^\tau = (1 - \tau)\mathcal{L} + \tau\mathcal{L}^*$ for $\tau \in [0, 1]$. We shall prove that the family of functionals $\{\mathcal{L}^\tau \mid \tau \in [0, 1]\}$ on $\Lambda_N(M)$ satisfies the stability property of critical groups [16, 18, 15, 19, 21, 36], and so

$$C_q(\mathcal{L}, \gamma_0; \mathbb{K}) \cong C_q(\mathcal{L}^*, \gamma_0; \mathbb{K}) \quad \forall q \geq 0.$$

By Corollary A.2 we have a shifting theorem for $C_q(\mathcal{L}^*, \gamma_0; \mathbb{K})$ and hence arrive at the desired results.

Our modified Lagrangian L^* can be required to be no more than L . This is very key for the proofs of Theorem 1.5(v), Theorem 1.8(i) and Theorem 1.9.

Organization of the paper. In Section 2 we start from $L = F^2$ to construct a suitable convex quadratic growth Lagrangian L^* having the properties outlined above. Then by considering the corresponding functional family $(\mathcal{L}^\tau)_{\tau \in [0, 1]}$ with the Lagrangians $L^\tau = (1 - \tau)L + \tau L^*$ with $\tau \in [0, 1]$ we show in Section 3 that the

functionals \mathcal{L} and \mathcal{L}^* have the same critical groups at γ_0 and $S^1 \cdot \gamma_0$ in two cases, respectively. The proofs of Theorems 1.2, 1.3 are given in Section 4, and those of Theorems 1.5, 1.6, 1.7 are given in Section 5. Section 6 deals with critical groups of iterated closed geodesics, including the proofs of Theorems 1.9, 1.10. In Section 7 we present a computation method of S^1 -critical groups. The proof of Theorem 1.11 is given in Section 8. Finally, in Appendixes A, B we state the shifting theorem obtained in [32, 34] and give some related computations, respectively.

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2 The modifications for the energy functionals

We firstly construct two smooth functions, see Figure 1.

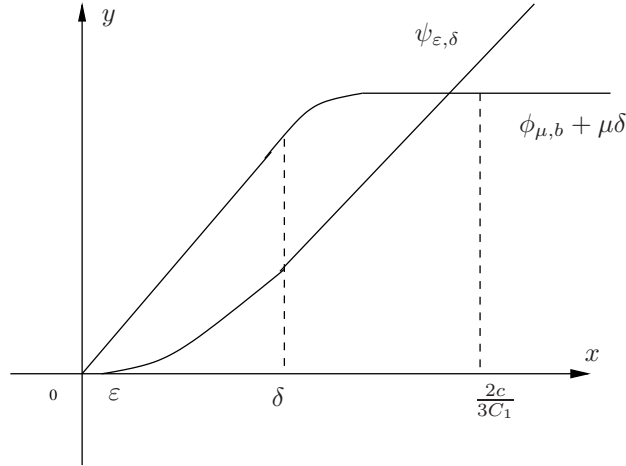


Figure 1: The functions $\psi_{\varepsilon,\delta}$ and $\phi_{\mu,b} + \mu\delta$.

Lemma 2.1 *Given positive numbers $c > 0$ and $C_1 \geq 1$, choose positive parameters $0 < \varepsilon < \delta < \frac{2c}{3C_1}$. Then*

- (i) *there exists a C^∞ function $\psi_{\varepsilon,\delta} : [0, \infty) \rightarrow \mathbb{R}$ such that: $\psi'_{\varepsilon,\delta} > 0$ and convex on (ε, ∞) , vanishes in $[0, \varepsilon)$, is equal to the affine function $\kappa t + \varrho_0$ on $[\delta, \infty)$, where $\kappa > 0$ and $\varrho_0 < 0$ are suitable constants;*

- (ii) there exists a C^∞ function $\phi_{\mu,b} : [0, \infty) \rightarrow \mathbb{R}$ depending on parameters $\mu > 0$ and $b > 0$, such that: $\phi_{\mu,b}$ is nondecreasing and concave (and hence $\phi_{\mu,b}'' \leq 0$), is equal to the affine function $\mu t - \mu\delta$ on $[0, \delta]$, is equal to constant $b > 0$ on $[\frac{2c}{3C_1}, \infty)$.
- (iii) Under the above assumptions, $\psi_{\varepsilon,\delta}(t) + \phi_{\mu,b}(t) - b = \kappa t + \varrho_0$ for any $t \geq \frac{2c}{3C_1}$ (and hence $t \geq \frac{2c}{3}$). Moreover, $\psi_{\varepsilon,\delta}(t) + \phi_{\mu,b}(t) - b \geq -\mu\delta - b \ \forall t \geq 0$, and $\psi_{\varepsilon,\delta}(t) + \phi_{\mu,b}(t) - b = -\mu\delta - b$ if and only if $t = 0$.
- (iv) Under the assumptions (i)-(ii), suppose that the constant $\mu > 0$ satisfies

$$\mu + \frac{\varrho_0}{\delta - \varepsilon} > 0 \quad \text{and} \quad \mu\delta + b + \varrho_0 > 0. \quad (2.1)$$

Then $\psi_{\varepsilon,\delta}(t) + \phi_{\mu,b}(t) - b \leq \kappa t + \varrho_0 \ \forall t \geq \varepsilon$, and $\psi_{\varepsilon,\delta}(t) + \phi_{\mu,b}(t) - b \leq \kappa t + \varrho_0 \ \forall t \in [0, \varepsilon]$ if $\kappa \geq \mu$.

Proof of Lemma 2.1. (i)-(iii) are easily obtained. We only prove (iv). Obverse that the line connecting points $(\varepsilon, 0)$ to $(\delta, \kappa\delta + \varrho_0)$ given by

$$\mathbb{R} \ni t \mapsto \frac{\kappa\delta + \varrho_0}{\delta - \varepsilon}(t - \varepsilon),$$

satisfies

$$\kappa t + \varrho_0 \leq \psi_{\varepsilon,\delta}(t) \leq \frac{\kappa\delta + \varrho_0}{\delta - \varepsilon}(t - \varepsilon) \quad \forall t \in [\varepsilon, \delta]. \quad (2.2)$$

Since $\phi_{\mu,b}(t) - b \leq 0 \ \forall t \geq 0$ we have $\psi_{\varepsilon,\delta}(t) + \phi_{\mu,b}(t) - b \leq \kappa t + \varrho_0 \ \forall t \in [\delta, \infty)$ by the definition of $\psi_{\varepsilon,\delta}$. When $t \in [\varepsilon, \delta]$, since $\phi_{\mu,b}(t) = \mu t - \mu\delta$ for any $t \leq \delta$, by (2.2) we only need to prove

$$\frac{\kappa\delta + \varrho_0}{\delta - \varepsilon}(t - \varepsilon) + \mu t - \mu\delta - b \leq \kappa t + \varrho_0 \quad \forall t \in [\varepsilon, \delta]. \quad (2.3)$$

Clearly, this is equivalent to

$$\left(\frac{\kappa\varepsilon + \varrho_0}{\delta - \varepsilon} + \mu \right) t \leq \varrho_0 + \mu\delta + b + \frac{\kappa\delta + \varrho_0}{\delta - \varepsilon}\varepsilon \quad \forall t \in [\varepsilon, \delta] \quad (2.4)$$

because $\frac{\kappa\delta}{\delta - \varepsilon} - \kappa = \frac{\kappa\varepsilon}{\delta - \varepsilon}$. The first inequality in (2.1) implies $\frac{\kappa\varepsilon + \varrho_0}{\delta - \varepsilon} + \mu \geq \frac{\kappa\varepsilon}{\delta - \varepsilon} > 0$. Hence it suffices to prove that (2.4), or equivalently, (2.3) holds for $t = \delta$, that is,

$$\frac{\kappa\delta + \varrho_0}{\delta - \varepsilon}(\delta - \varepsilon) + \mu\delta - \mu\delta - b \leq \kappa\delta + \varrho_0.$$

It is obvious because $b > 0$. This proves the first claim in (iv).

For the case $t \in [0, \varepsilon]$ we have $\psi_{\varepsilon,\delta}(t) = 0$ and

$$\begin{aligned} \phi_{\mu,b}(t) - b &= \mu t - \mu\delta - b \\ &\leq \kappa t + \mu\delta + b + \varrho_0 - \mu\delta - b \\ &= \kappa t + \varrho_0 \end{aligned}$$

because the second inequality in (2.1) implies $(\mu - \kappa)t \leq \mu\delta + b + \varrho_0$ for any $\mu < \kappa$. Lemma 2.1(iv) is proved. \square

Proposition 2.2 *Let (M, F) be a C^k Finsler manifold ($k \geq 2$), and $L := F^2$. Suppose that for some Riemannian metric h on M both*

$$\alpha_h := \inf_{h_x(v,v)=1} \inf_{u \neq 0} \frac{g(x,v)[u,u]}{h_x(u,u)} \quad \text{and} \quad \beta_h := \sup_{h_x(v,v)=1} \sup_{u \neq 0} \frac{g(x,v)[u,u]}{h_x(u,u)}$$

are positive numbers, and that for some constant $C_1 > 0$,

$$|v|_x^2 \leq L(x, v) \leq C_1 |v|_x^2 \quad \forall (x, v) \in TM. \quad (2.5)$$

Hereafter $|v|_x = \sqrt{h_x(v, v)}$ without special statements. Then for a given $c > 0$ there exists a C^k Lagrangian $L^ : TM \rightarrow \mathbb{R}$ such that*

- (i) $L^*(x, v) = \kappa L(x, v) + \varrho$ if $L(x, v) \geq \frac{2c}{3C_1}$,
- (ii) L^* attains the minimum, and $L^*(x, v) = \min L^* \iff v = 0$,
- (iii) $L^*(x, v) \leq \kappa L(x, v) + \varrho_0$ for all $(x, v) \in TM$,
- (iv) $\partial_{vv} L^*(x, v)[u, u] \geq \min\{2\mu, \frac{1}{2}\kappa\alpha_h\}|u|_x^2$. (This implies (ii).)

Moreover, if F is reversible, i.e. $F(x, -v) = F(x, v) \forall (x, v) \in TM$, so is L^ .*

Proof. By the assumptions we have

$$\alpha_h |u|_x^2 \leq g(x, v)[u, u] \leq \beta_h |u|_x^2 \quad (2.6)$$

for all $(x, v) \in TM \setminus \{0\}$ and $(x, u) \in TM$.

Let $\psi_{\varepsilon, \delta}$ and $\phi_{\mu, b}$ be as in Lemma 2.1. Suppose that (2.1) is satisfied and that $\kappa \geq \mu$. Consider the function $L^* : TM \rightarrow \mathbb{R}$ defined by

$$L^*(x, v) = \psi_{\varepsilon, \delta}(L(x, v)) + \phi_{\mu, b}(|v|_x^2) - b. \quad (2.7)$$

Clearly, it is C^k smooth, and satisfies the final claim.

By (2.5), $\phi_{\mu, b}(|v|_x^2) \leq \phi_{\mu, b}(L(x, v)) \forall (x, v)$. (i) and (ii) follows from Lemma 2.1(iii) immediately. Obverse that

$$L^*(x, v) = \psi_{\varepsilon, \delta}(L(x, v)) + \phi_{\mu, b}(|v|_x^2) - b \leq \psi_{\varepsilon, \delta}(L(x, v)) + \phi_{\mu, b}(L(x, v)) - b$$

for any (x, v) . We derive (iii) from Lemma 2.1(iv).

Now we are a position to prove (iv). Since

$$\begin{aligned} \frac{\partial^2}{\partial t \partial s} \psi_{\varepsilon, \delta}(L(x, v + su + tu)) &= \frac{\partial}{\partial t} \left[\psi'_{\varepsilon, \delta}(L(x, v + su + tu)) \frac{\partial}{\partial s} L(x, v + su + tu) \right] \\ &= \psi'_{\varepsilon, \delta}(L(x, v + su + tu)) \frac{\partial^2}{\partial t \partial s} L(x, v + su + tu) \\ &\quad + \psi''_{\varepsilon, \delta}(L(x, v + su + tu)) \frac{\partial}{\partial s} L(x, v + su + tu) \frac{\partial}{\partial t} L(x, v + su + tu) \end{aligned}$$

for $u \in T_x M$ and $v \in T_x M \setminus \{0\}$, we get

$$\begin{aligned} &\frac{\partial^2}{\partial t \partial s} \psi_{\varepsilon, \delta}(L(x, v + su + tu)) \Big|_{s=0, t=0} \\ &= \psi'_{\varepsilon, \delta}(L(x, v)) \partial_{vv}^2 L(x, v)[u, u] + \psi''_{\varepsilon, \delta}(L(x, v)) [\partial_v L(x, v)(u)]^2. \end{aligned}$$

Clearly, the left side is equal to zero at $v = 0$. Similarity, one easily computes

$$\frac{\partial^2}{\partial t \partial s} \phi_{\mu,b}(|v + su + tu|_x^2) \Big|_{s=0, t=0} = 4\phi''_{\mu,b}(|v|_x^2) \langle v, u \rangle_x^2 + 2\phi'_{\mu,b}(|v|_x^2) |u|_x^2.$$

These lead to

$$\begin{aligned} \partial_{vv} L^*(x, v)[u, u] &= \psi''_{\varepsilon, \delta}(L(x, v)) [\partial_v L(x, v)(u)]^2 + \psi'_{\varepsilon, \delta}(L(x, v)) \partial_{vv}^2 L(x, v)(u, u) \\ &+ 4\eta \phi''_{\mu,b}(|v|_x^2) \langle v, u \rangle_x^2 + 2\eta \phi'_{\mu,b}(|v|_x^2) |u|_x^2. \end{aligned} \quad (2.8)$$

- If $L(x, v) \leq \delta$ (and hence $|v|_x^2 \leq \delta$ by (2.5)), then

$$\partial_{vv} L^*(x, v)[u, u] \geq 2\mu\eta |u|_x^2 \quad (2.9)$$

because Lemma 2.1(i) and Lemma 2.1(ii) imply: $\psi''_{\varepsilon, \delta}(L(x, v)) \geq 0$, $\psi'_{\varepsilon, \delta}(L(x, v)) \geq 0$, and $\phi''_{\mu,b}(|v|_x^2) = 0$ and $\phi'_{\mu,b}(|v|_x^2) = \mu$ for $|v|_x^2 \leq \delta$.

- If $L(x, v) \geq \delta$ (and hence $|v|_x^2 \geq \frac{\delta}{C_1} > \frac{\delta}{3C_1}$ by (2.5)), then $\psi''_{\varepsilon, \delta}(L(x, v)) = 0$ and

$$\psi'_{\varepsilon, \delta}(L(x, v)) \partial_{vv} L(x, v)(u, u) = \kappa \partial_{vv} L(x, v)(u, u)$$

by Lemma 2.1(i). Hence it follows from (2.8), Lemma 2.1(ii) and (2.6) that

$$\begin{aligned} \partial_{vv} L^*(x, v)[u, u] &\geq \kappa \partial_{vv} L(x, v)(u, u) + 4\eta \phi''_{\mu,b}(|v|_x^2) \langle v, u \rangle_x^2 + 2\eta \phi'_{\mu,b}(|v|_x^2) |u|_x^2 \\ &\geq \kappa \partial_{vv} L(x, v)(u, u) + 4\eta \phi''_{\mu,b}(|v|_x^2) \langle v, u \rangle_x^2 \\ &\geq \kappa \alpha_h |u|_x^2 + 4\eta \phi''_{\mu,b}(|v|_x^2) |v|_x^2 |u|_x^2 \\ &\geq \kappa \alpha_h |u|_x^2 + \frac{8c}{3C_1} \phi''_{\mu,b}(|v|_x^2) |u|_x^2 \\ &= \left[\kappa \alpha_h + \frac{8c}{3C_1} \phi''_{\mu,b}(|v|_x^2) \right] |u|_x^2 \end{aligned}$$

because $\phi''_{\mu,b} \leq 0$, and $\phi''_{\mu,b}(|v|_x^2) = 0$ for $|v|_x^2 \geq \frac{2c}{3C_1}$. Obverse that $\phi''_{\mu,b}(|v|_x^2)$ is bounded for $|v|_x^2 \in [\frac{\delta}{3C_1}, \frac{2c}{3C_1}]$. We may choose $\kappa > 0$ so large that

$$\kappa \alpha_h + \frac{8c}{3C_1} \phi''_{\mu,b}(|v|_x^2) \geq \frac{1}{2} \kappa \alpha_h.$$

This and (2.9) yield the expected conclusion. \square

Proposition 2.2(iii) and thus the following Corollary 2.3(iii) is only used in the proof of Theorem 6.1. (i)-(iii) in Lemma 2.1 and large $\kappa > 0$ are sufficient for other arguments.

If M is compact, for any Riemannian metric h on M both α_h and β_h are positive numbers, and they may be chosen so that (2.5) holds for some constant $C_1 > 0$. In this case it is easily proved that L^* in (2.7) is a convex quadratic growth Lagrangian. Defining $L^*(x, v)$ by $(L^*(x, v) - \varrho_0)/\kappa$ we get

Corollary 2.3 *Let (M, F) be a compact C^k Finsler manifold ($k \geq 2$), and $L := F^2$. Then for a given $c > 0$ there exists a C^k convex quadratic growth Lagrangian $L^* : TM \rightarrow \mathbb{R}$ such that*

- (i) $L^*(x, v) = L(x, v)$ if $L(x, v) \geq \frac{2c}{3C_1}$,
- (ii) L^* attains the minimum, and $L^*(x, v) = \min L^* \iff v = 0$,
- (iii) $L^*(x, v) \leq L(x, v)$ for all $(x, v) \in TM$,
- (iv) if F is reversible, so is L^* .

(ii) is actually implied in the convexity of L^* . We can also construct another C^k convex quadratic growth Lagrangian $L^* : TM \rightarrow \mathbb{R}$ satisfying Corollary 2.3(i)(ii)(iv) and the condition $L^*(x, v) \geq L(x, v)$ for all $(x, v) \in TM$. Such a L^* is not needed in this paper.

Let L^* be as in Corollary 2.3. By the compactness of M , (2.5)-(2.6) and the fact that L^* is convex quadratic growth we may derive that there exist $\alpha_h^* > 0$, $\beta_h^* > 0$, $C_2 > 0$ and $C_3 > 0$ such that

$$\alpha_h^* |u|_x^2 \leq \partial_{vv} L^*(x, v)[u, u] \leq \beta_h^* |u|_x^2, \quad (2.10)$$

$$C_2 |v|_x^2 \leq L^*(x, v) \leq C_3 (|v|_x^2 + 1). \quad (2.11)$$

For each $\tau \in [0, 1]$ we define $L^\tau : TM \rightarrow \mathbb{R}$ by

$$L^\tau(x, v) = (1 - \tau)L(x, v) + \tau L^*(x, v). \quad (2.12)$$

Then it is only C^k in $TM \setminus \{0\}$ for $0 \leq \tau < 1$, and by (2.5)-(2.6) and (2.10)-(2.11) ones easily prove that in $TM \setminus \{0\}$,

$$\min\{\alpha_h, \alpha_h^*\} |u|_x^2 \leq \partial_{vv} L^\tau(x, v)[u, u] \leq \max\{\beta_h, \beta_h^*\} |u|_x^2, \quad (2.13)$$

$$\min\{C_2, 1\} |v|_x^2 \leq L^\tau(x, v) \leq (C_1 + C_3) (|v|_x^2 + 1). \quad (2.14)$$

From the mean value theorem it follows that

$$\begin{aligned} |\partial_v L^\tau(x, v) \cdot v| &= |\partial_v L^\tau(x, v) \cdot v - \partial_v L^\tau(x, 0) \cdot v| \\ &= \left| \int_0^1 \partial_{vv} L^\tau(x, sv)[v, v] ds \right| \\ &\geq \min\{\alpha_h, \alpha_h^*\} |v|_x^2 \end{aligned}$$

for any $(x, v) \in TM \setminus \{0\}$, and therefore

$$|\partial_v L^\tau(x, v)| \geq \min\{\alpha_h, \alpha_h^*\} |v|_x \quad \forall (x, v) \in TM. \quad (2.15)$$

Consider the Legendre transform associated with L^τ ,

$$\mathbf{L}^\tau : TM \rightarrow T^*M, (x, v) \mapsto (x, \partial_v L^\tau(x, v)). \quad (2.16)$$

Proposition 2.4 \mathbf{L}^τ is a homeomorphism, and a C^{k-1} -diffeomorphism for $\tau = 1$. Moreover, \mathbf{L}^τ also restricts a C^{k-1} -diffeomorphism from $TM \setminus \{0\}$ onto $T^*M \setminus \{0\}$.

Proof. The conclusion for $\tau = 1$ is standard (see [23, Prop.2.1.6]). The case $\tau = 0$ had been proved in the proof of [13, Prop.2.1]. The similar proof yields the case $0 < \tau < 1$. That is, we only need to prove that they are proper local homeomorphisms by Banach-Mazur theorem (cf. [5, Th.5.1.4]).

Since L^τ is C^1 , fiberwise strictly convex and superlinear on TM the map

$$\mathbf{L}_x^\tau : T_x M \rightarrow T_x^* M, v \mapsto \partial_v L^\tau(x, v)$$

is a homeomorphism by Theorem 1.4.6 of [23]. Moreover, L^τ is C^k on $TM \setminus \{0\}$, and $\partial_{vv} L^\tau(x, v)[u, u] > 0$ for any $u, v \in T_x M \setminus \{0\}$. From the implicit function theorem it follows that the map

$$\mathbf{L}_x^\tau : T_x M \setminus \{0_x\} \ni v \mapsto \partial_v L^\tau(x, v) \in T_x^* M \setminus \{0_x\}$$

is a C^{k-1} diffeomorphism.

Now $\mathbf{L}^\tau : TM \rightarrow T^*M$ is a continuous bijection. Consider its inverse

$$(\mathbf{L}^\tau)^{-1} : TM \rightarrow T^*M, (x, v) \mapsto (x, (\mathbf{L}_x^\tau)^{-1}v).$$

By the positivity of $\partial_{vv} L^\tau$ on $TM \setminus \{0\}$, from the inverse function theorem we derive that \mathbf{L}^τ is locally a C^{k-1} diffeomorphism on $TM \setminus \{0\}$ and $(\mathbf{L}^\tau)^{-1}$ maps $T^*M \setminus \{0\}$ onto $TM \setminus \{0\}$.

We claim that the continuity of $(\mathbf{L}^\tau)^{-1}$ extends up to the zero section. Suppose that $(x_n, w_n) \rightarrow (x_0, 0)$ and

$$(\mathbf{L}^\tau)^{-1}(x_n, w_n) = (x_n, v_n).$$

Then $\mathbf{L}^\tau(x_n, v_n) = (x_n, w_n)$ or $w_n = \partial_v L^\tau(x_n, v_n)$. By (2.15) we deduce that $|v_n|_{x_n} \rightarrow 0$. This leads to the desired claim. Hence \mathbf{L}^τ is a homeomorphism from TM onto T^*M . The second conclusion is a direct consequence of this fact and the inverse function theorem. \square

Remark 2.5 For a C^k convex quadratic growth Lagrangian $L^* : TM \rightarrow \mathbb{R}$ ($k \geq 2$), by the convexity of L^* we have $\partial_v L^*(x, 0) = 0 \forall x$ and therefore (2.15) holds, it follows that Proposition 2.4 is still true.

3 The stability of critical groups

In this section we assume that (M, F) is a compact C^k Finsler manifold ($k \geq 2$) and $L := F^2$. For a C^k convex quadratic growth Lagrangian $L^* : TM \rightarrow \mathbb{R}$ and $\tau \in [0, 1]$ we define $L^\tau(x, v) = (1 - \tau)L(x, v) + \tau L^*(x, v)$ and

$$\mathcal{L}^\tau(\gamma) = \int_0^1 L^\tau(\gamma(t), \dot{\gamma}(t)) dt \quad \forall \gamma \in \Lambda_N(M). \quad (3.1)$$

Then $\mathcal{L}^\tau(\gamma) = (1 - \tau)\mathcal{L}(\gamma) + \tau\mathcal{L}^*(\gamma)$ for $\tau \in [0, 1]$, where the functionals \mathcal{L} and \mathcal{L}^* are given by (1.2) and (1.24), respectively. Since \mathcal{L} and \mathcal{L}^* on $\Lambda_N(M)$ are C^{2-0} and

satisfy the Palais-Smale condition by Proposition 1.1 and [1], so is each functional \mathcal{L}^τ on $\Lambda_N(M)$.

The **energy function** of L^τ , $E^\tau : TM \rightarrow \mathbb{R}$ is defined by

$$E^\tau(x, v) = \partial_v L^\tau(x, v) \cdot v - L^\tau(x, v). \quad (3.2)$$

(It is C^{1-0} because F^2 is C^{2-0} on TM).

Proposition 3.1 *For the Lagrangian L^* in Corollary 2.3, if $\gamma \in \Lambda_N(M)$ is a critical point of the functional \mathcal{L}^τ which is not a constant curve, then γ is a C^k regular curve, i.e. $\dot{\gamma}(t) \neq 0$ for any $t \in I$.*

Proof. Fix a point $\gamma(\bar{t})$ and choose a coordinate chart around $\gamma(\bar{t})$ on M , (V, χ) ,

$$\chi : V \rightarrow \chi(V) \subset \mathbb{R}^n, \quad x \mapsto \chi(x) = (x_1, \dots, x_n).$$

Then we get an induced chart on TM , $(\pi^{-1}(V), T\chi)$,

$$T\chi : \pi^{-1}(V) \rightarrow \chi(V) \times \mathbb{R}^n, \quad (x, v) \mapsto (x_1, \dots, x_n; v_1, \dots, v_n).$$

Let I_0 be a connected component of $\gamma^{-1}(V)$. It has one of the following three forms: $[0, a)$, (a, b) , $(b, 1]$. Let $\tilde{\gamma}(t) := \chi(\gamma(t))$ for $t \in I_0$. Then $\tilde{\gamma} : I_0 \rightarrow V \subset \mathbb{R}^n$ is absolutely continuous and $\dot{\tilde{\gamma}}(t) = d\chi(\gamma(t))(\dot{\gamma}(t))$ for $t \in I_0$. Set

$$\tilde{L}^\tau(x, y) := L^\tau(\chi^{-1}(x), d\chi^{-1}(x)(y)) \quad \forall (x, y) \in \chi(V) \times \mathbb{R}^n.$$

Since $d\mathcal{L}^\tau(\gamma)(\xi) = 0 \quad \forall \xi \in T_\gamma \Lambda_N(M)$, we deduce that

$$\int_{I_0} \left(\partial_x \tilde{L}^\tau(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t))[\xi(t)] + \partial_y \tilde{L}^\tau(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t))[\dot{\xi}(t)] \right) dt = 0$$

for any $\xi \in C^\infty(I_0, \mathbb{R}^n)$ with $\text{supp}(\xi) \subset \text{Int}(I_0)$. Denote by t_0 the left end point of I_0 and by

$$H(t) = - \int_{t_0}^t \partial_x \tilde{L}^\tau(\tilde{\gamma}(s), \dot{\tilde{\gamma}}(s)) ds \quad \forall t \in I_0.$$

Then $I_0 \ni t \mapsto H(t) \in \mathbb{R}^n$ is continuous and

$$\int_{I_0} \left(H(t) + \partial_y \tilde{L}^\tau(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) \right) [\dot{\xi}(t)] dt = 0$$

for any $\xi \in C^\infty(I_0, \mathbb{R}^n)$ with $\text{supp}(\xi) \subset \text{Int}(I_0)$. It follows that there exists a constant vector $\mathbf{u} \in \mathbb{R}^n$ such that

$$H(t) + \partial_y \tilde{L}^\tau(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) = \mathbf{u} \quad \text{a.e. on } I_0.$$

This implies the map

$$I_0 \ni t \mapsto \partial_y \tilde{L}^\tau(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t))$$

is continuous. Moreover, as in Proposition 2.4 we can prove that the map

$$\tilde{\mathbf{L}}^\tau : \chi(V) \times \mathbb{R}^n \rightarrow \chi(V) \times \mathbb{R}^n, \quad (x, y) \mapsto (x, \partial_y \tilde{L}^\tau(x, y))$$

is a homeomorphism, and also restricts to a C^{k-1} -diffeomorphism $\tilde{\mathbf{L}}_0^\tau$ from $\chi(V) \times (\mathbb{R}^n \setminus \{0\})$ to $\chi(V) \times (\mathbb{R}^n \setminus \{0\})$. So

$$I_0 \ni t \mapsto (\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) = (\tilde{\mathbf{L}}^\tau)^{-1} \circ \tilde{\mathbf{L}}^\tau(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) = (\tilde{\mathbf{L}}^\tau)^{-1}(\tilde{\gamma}(t), \partial_y \tilde{L}^\tau(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)))$$

is continuous. This shows that $\tilde{\gamma}$ is C^1 . Obverse that

$$\frac{d}{dt} \partial_y \tilde{L}^\tau(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) = \partial_x \tilde{L}^\tau(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) \quad \text{a.e. on } I_0.$$

We get that the map $I_0 \ni t \mapsto \partial_y \tilde{L}^\tau(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t))$ is C^1 , and hence the composition

$$\{t \in I_0 \mid \dot{\gamma}(t) \neq 0\} \ni t \mapsto \partial_y \tilde{L}^\tau(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) \xrightarrow{(\tilde{\mathbf{L}}^\tau)^{-1}} (\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t))$$

is C^1 . In summary we prove: **γ is C^1 , and C^2 in $\{t \in I \mid \dot{\gamma}(t) \neq 0\}$.**

Now since γ is not constant. Then the energy

$$E^\tau(\gamma, \dot{\gamma}) = \partial_v L^\tau(\gamma, \dot{\gamma}) \cdot \dot{\gamma} - L^\tau(\gamma, \dot{\gamma})$$

is constant on every connected component of $\{t \in I \mid \dot{\gamma}(t) \neq 0\}$. **Note** that Corollary 2.3 implies

$$L^\tau \geq \tau \min L^*, \quad L^\tau(x, 0) = \tau \min L^* \quad \text{and} \quad E^\tau(x, 0) = -\tau \min L^* \quad \forall x. \quad (3.3)$$

For $v \in T_x M \setminus \{0\}$ and $t > 0$, by (3.2) we have

$$\begin{aligned} \frac{d}{dt} E^\tau(x, tv) &= \frac{d}{dt} (\partial_v L^\tau(x, tv) \cdot (tv)) - \frac{d}{dt} L^\tau(x, tv) \\ &= \partial_{vv} L^\tau(x, tv)[v, tv] + \partial_v L^\tau(x, tv) \cdot v - \partial_v L^\tau(x, tv) \cdot v \\ &= t \partial_{vv} L^\tau(x, tv)[v, v] > 0. \end{aligned}$$

It follows that $E^\tau(x, v) > E^\tau(x, 0) = -\tau \min L^*$ on $TM \setminus \{0\}$ and hence

$$E^\tau(x, v) \geq -\tau \min L^* \quad \text{and} \quad E^\tau(x, v) = -\tau \min L^* \iff v = 0.$$

Since $E^\tau(\gamma, \dot{\gamma})$ is strictly larger than $-\tau \min L^*$ in $I \setminus \{t \in I \mid \dot{\gamma}(t) \neq 0\}$ and $I \ni t \rightarrow E^\tau(\gamma(t), \dot{\gamma}(t))$ is continuous we must have $I = \{t \in I \mid \dot{\gamma}(t) \neq 0\}$. It easily follows that γ is C^k . \square

Remark 3.2 For a C^k convex quadratic growth Lagrangian $L^* : TM \rightarrow \mathbb{R}$ ($k \geq 2$), if $L^*(x, 0) = \min L^* \forall x$ then (2.15) and therefore Proposition 2.4 hold. Moreover we have also (3.3) and hence Proposition 3.1.

Since the weak slope of a C^1 functional f on an open subset of a normed space is equal to the norm of the differential of f the lower critical point (resp. value) of f agrees with the usual one of f . The following special version of [19, Th.1.5] about the stability property of critical groups is convenient for us.

Theorem 3.3 ([19, Th.1.5]) *Let $\{f_\tau : \tau \in [0, 1]\}$ be a family of C^1 functionals from a Banach space X to \mathbb{R} , U an open subset of X and $[0, 1] \ni \tau \mapsto u_\tau \in U$ a continuous map. Assume:*

- (I) *if $\tau_k \rightarrow \tau$ in $[0, 1]$, then $f_{\tau_k} \rightarrow f_\tau$ uniformly on \overline{U} ;*
- (II) *for every sequence $\tau_k \rightarrow \tau$ in $[0, 1]$ and (v_k) in \overline{U} with $f'_{\tau_k}(v_k) \rightarrow 0$ and $(f_{\tau_k}(v_k))$ bounded, there exists a subsequence (v_{k_j}) convergent to some v with $f'_\tau(v) = 0$;*
- (III) *$f'_\tau(u) > 0$ for every $\tau \in [0, 1]$ and $u \in \overline{U} \setminus \{u_\tau\}$*

Then $C_q(f_0, u_0; \mathbb{K}) \cong C_q(f_1, u_1; \mathbb{K})$ for every $q \geq 0$.

Actually, Chang [16, page 53, Th.5.6], Chang and Ghoussoub [18, Th.III.4] and Corvellec and Hantoute [21, Th.5.2] are sufficient for the proof of our Theorem 3.8.

For $\sigma, \tau \in [0, 1]$, since $L^\sigma - L^\tau = (\tau - \sigma)L + (\sigma - \tau)L^*$ we have

$$|\mathcal{L}^\sigma(\gamma) - \mathcal{L}^\tau(\gamma)| \leq |\sigma - \tau| \int_0^1 \left[|L(\gamma(t), \dot{\gamma}(t))| + |L^*(\gamma(t), \dot{\gamma}(t))| \right] dt \quad (3.4)$$

for all $\gamma \in \Lambda_N(M)$. Note that the condition (L2) and the compactness of M implies

$$|L^*(x, v)| \leq C_2(1 + |v|_x^2) \quad \forall (x, v) \in TM \quad (3.5)$$

for some constant $C_2 > 0$. Let $C_3 := \max\{L(x, v) \mid |v|_x = 1\}$. Then

$$|L(x, v)| \leq C_3|v|_x^2 \quad \forall (x, v) \in TM.$$

From this and (3.4)-(3.5) we immediately get

Claim 3.4 *For any bounded subset $K \subset \Lambda_N(M)$, $\mathcal{L}^\sigma \rightarrow \mathcal{L}^\tau$ uniformly on K as $\sigma \rightarrow \tau$.*

Claim 3.5 *For any $\gamma_0 \in \Lambda_N(M)$ there exists a neighborhood \mathcal{U} of it such that for every sequence $\tau_k \rightarrow \tau$ in $[0, 1]$ and a sequence (γ_k) in \mathcal{U} with $d\mathcal{L}^{\tau_k}(\gamma_k) \rightarrow 0$ and $(\mathcal{L}^{\tau_k}(\gamma_k))$ bounded, there exists a subsequence (γ_{k_j}) convergent to some γ with $d\mathcal{L}^\tau(\gamma) = 0$. (Clearly, γ_0 can be replaced by a compact subset $K \subset \Lambda_N(M)$.)*

Proof. Clearly, this result is of a local nature. By a well-known localization argument as in [31] (cf. Section 4 below) the question is reduced to the following case:

- $F, L^* : B_{2\rho}^n(0) \times \mathbb{R}^n \rightarrow \mathbb{R}$;
- $(\gamma_k) \subset W_V^{1,2}([0, 1], B_\rho^n(0)) := \{\gamma \in W^{1,2}([0, 1], B_\rho^n(0)) \mid (\gamma(0), \gamma(1)) \in V\}$ is bounded, where V is a linear subspace of $\mathbb{R}^n \times \mathbb{R}^n$, such that

$$d\mathcal{L}^{\tau_k}(\gamma_k) \rightarrow 0 \text{ (as } k \rightarrow \infty) \quad \text{and} \quad |\mathcal{L}^{\tau_k}(\gamma_k)| \leq C_4 \quad \forall k \quad (3.6)$$

for some constant C_4 and that both $(\|d\mathcal{L}(\gamma_k)\|)$ and $(\|d\mathcal{L}^*(\gamma_k)\|)$ are bounded;

- The conclusion is that there exists a subsequence (γ_{k_j}) convergent to some $\gamma \in W_V^{1,2}([0, 1], B_{2\rho}^n(0))$ with $d\mathcal{L}_\tau(\gamma) = 0$. In this time γ satisfies

$$\begin{aligned} \frac{d}{dt}(\partial_v L^\tau(\gamma(t), \dot{\gamma}(t))) - \partial_x L^\tau(\gamma(t), \dot{\gamma}(t)) &= 0, \\ \partial_v L^\tau(\gamma(0), \dot{\gamma}(0)) \cdot v_0 &= \partial_v L^\tau(\gamma(1), \dot{\gamma}(1)) \cdot v_1 \quad \forall (v_0, v_1) \in V. \end{aligned}$$

In order to prove this, note that (3.6) implies

$$\begin{aligned}\|d\mathcal{L}^\tau(\gamma_k)\| &\leq \|d\mathcal{L}^{\tau_k}(\gamma_k)\| + \|d\mathcal{L}^\tau(\gamma_k) - d\mathcal{L}^{\tau_k}(\gamma_k)\| \\ &\leq \|d\mathcal{L}^{\tau_k}(\gamma_k)\| + |\tau_k - \tau| \cdot (\|d\mathcal{L}(\gamma_k)\| + \|d\mathcal{L}^*(\gamma_k)\|).\end{aligned}$$

We arrive at

$$\|d\mathcal{L}^\tau(\gamma_k)\| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Moreover Claim 3.4 implies that $(|\mathcal{L}^\tau(\gamma_k)|)$ is bounded too. Combing the proof of [12, Th.3.1] and that of [1, Th.3.1(iv)] we can complete the proof of Claim 3.5. They are omitted. \square

Claim 3.6 *For any $C > 0$ there exists a $C' > 0$, which is independent of $\tau \in [0, 1]$ and $(x, v) \in TM$, such that*

$$E^\tau(x, v) \leq C \implies |v|_x \leq C'.$$

Proof. Since $L^*(x, v) = (\psi_{\varepsilon, \delta}(L(x, v)) + \phi_{\mu, b}(|v|_x^2) - b - \varrho_0)/\kappa$, we have

$$\begin{aligned}E^*(x, v) &= \partial_v L^*(x, v) \cdot v - L^*(x, v) \\ &= \frac{1}{\kappa} [\psi'_{\varepsilon, \delta}(L(x, v)) \partial_v L(x, v) \cdot v + 2\phi'_{\mu, b}(|v|_x^2) |v|_x^2 \\ &\quad - \psi_{\varepsilon, \delta}(L(x, v)) - \phi_{\mu, b}(|v|_x^2) + b + \varrho_0].\end{aligned}$$

Moreover, $E(x, v) = \partial_v L(x, v) \cdot v - L(x, v) = L(x, v)$. Hence

$$\begin{aligned}E^\tau(x, v) &= (1 - \tau)L(x, v) + \frac{\tau}{\kappa} [b + \varrho_0 + 2\psi'_{\varepsilon, \delta}(L(x, v))L(x, v) \\ &\quad - \psi_{\varepsilon, \delta}(L(x, v)) + 2\phi'_{\mu, b}(|v|_x^2)|v|_x^2 - \phi_{\mu, b}(|v|_x^2)].\end{aligned}$$

Suppose that there exist sequences $(\tau_k) \subset [0, 1]$ with $\tau_k \rightarrow \tau$ and $(x_k, v_k) \subset TM$ with $x_k \rightarrow x_0$, such that

$$E^{\tau_k}(x_k, v_k) \leq C \quad \forall k, \quad \text{and} \quad |v_k|_{x_k} \rightarrow \infty.$$

Since (2.5) implies $L(x_k, v_k) \rightarrow \infty$ we deduce

$$\begin{aligned}\psi'_{\varepsilon, \delta}(L(x_k, v_k)) &= \kappa, \\ \psi_{\varepsilon, \delta}(L(x_k, v_k)) &= \kappa L(x_k, v_k) + \varrho_0, \\ \phi'_{\mu, b}(|v_k|_{x_k}^2) &= 0, \\ \phi_{\mu, b}(|v_k|_{x_k}^2) &= b\end{aligned}$$

for k sufficiently large. It follows that

$$\begin{aligned}E^{\tau_k}(x_k, v_k) &= (1 - \tau_k)L(x_k, v_k) + \frac{\tau_k}{\kappa} [2\kappa L(x_k, v_k) - \kappa L(x_k, v_k)] \\ &= L(x_k, v_k) \rightarrow \infty\end{aligned}$$

as $k \rightarrow \infty$. This contradiction yields the desired claim. \square

Claim 3.7 *Let $\gamma_0 \in \Lambda_N(M)$ be an isolated nonconstant critical point of \mathcal{L} on $\Lambda_N(M)$ (and hence a C^k nonconstant F -geodesics with constant speed $F(\gamma_0(t), \dot{\gamma}_0(t)) \equiv \sqrt{c} > 0$). Let $L^* : TM \rightarrow \mathbb{R}$ be given by Corollary 2.3, and let*

$$L^\tau(x, v) = (1 - \tau)L(x, v) + \tau L^*(x, v) \quad \forall \tau \in [0, 1].$$

Then there exists a neighborhood of γ_0 in $\Lambda_N(M)$, $\mathcal{U}(\gamma_0)$, such that each \mathcal{L}^τ has only a critical point γ_0 in $\mathcal{U}(\gamma_0)$.

Proof. Clearly, if $L(x, v) \geq \frac{2c}{3}$ then

$$L^\tau(x, v) = L(x, v). \quad (3.7)$$

Since $L(\gamma_0(t), \dot{\gamma}_0(t)) = c \forall t$, it is easily checked that γ_0 is a critical point of the functionals \mathcal{L}^τ in (3.1) on $\Lambda_N(M)$ and $L^\tau(\gamma_0(t), \dot{\gamma}_0(t)) = c \forall t$.

By a contradiction, suppose that there exist sequences $(\gamma_k) \subset \Lambda_N(M)$, $(\tau_k) \subset [0, 1]$ such that

$$\tau_k \rightarrow \tau_0, \quad \gamma_k \rightarrow \gamma_0, \quad d\mathcal{L}^{\tau_k}(\gamma_k) = 0 \quad \forall k. \quad (3.8)$$

Then γ_k is nonconstant for each large k , and therefore a C^k regular curve by Proposition 3.1 (removing finite many terms if necessary). Note that $E^{\tau_0}(\gamma_0(t), \gamma'_0(t))$ and $E^{\tau_k}(\gamma_k(t), \gamma'_k(t))$ are constants independent of t . Set $d_0 := E^{\tau_0}(\gamma_0(t), \gamma'_0(t))$ and $d_k := E^{\tau_k}(\gamma_k(t), \gamma'_k(t))$ for $k = 1, 2, \dots$. Then

$$\begin{aligned} d_0 &= \int_0^1 E^{\tau_0}(\gamma_0(t), \gamma'_0(t)) dt \\ &= \int_0^1 \partial_v L^{\tau_0}(\gamma_0(t), \gamma'_0(t))[\gamma'_0(t)] dt - \int_0^1 L^{\tau_0}(\gamma_0(t), \gamma'_0(t)) dt \\ &= (1 - \tau_0) \left[\int_0^1 \partial_v L(\gamma_0(t), \gamma'_0(t))[\gamma'_0(t)] dt - \int_0^1 L(\gamma_0(t), \gamma'_0(t)) dt \right] \\ &\quad + \tau_0 \left[\int_0^1 \partial_v L^*(\gamma_0(t), \gamma'_0(t))[\gamma'_0(t)] dt - \int_0^1 L^*(\gamma_0(t), \gamma'_0(t)) dt \right] \\ &= (1 - \tau_0) \int_0^1 L(\gamma_0(t), \gamma'_0(t)) dt \\ &\quad + \tau_0 \left[\int_0^1 \partial_v L^*(\gamma_0(t), \gamma'_0(t))[\gamma'_0(t)] dt - \int_0^1 L^*(\gamma_0(t), \gamma'_0(t)) dt \right] \end{aligned}$$

since $\partial_v L(x, v)[v] = 2L(x, v)$. Similarly, we have

$$\begin{aligned} d_k &= \int_0^1 E^{\tau_k}(\gamma_k(t), \gamma'_k(t)) dt \\ &= \int_0^1 \partial_v L^{\tau_k}(\gamma_k(t), \gamma'_k(t))[\gamma'_k(t)] dt - \int_0^1 L^{\tau_k}(\gamma_k(t), \gamma'_k(t)) dt \\ &= (1 - \tau_k) \int_0^1 L(\gamma_k(t), \gamma'_k(t)) dt \\ &\quad + \tau_k \left[\int_0^1 \partial_v L^*(\gamma_k(t), \gamma'_k(t))[\gamma'_k(t)] dt - \int_0^1 L^*(\gamma_k(t), \gamma'_k(t)) dt \right]. \end{aligned}$$

Recall that L^* is convex quadratic growth. There exists a constant $C^* > 0$ such that $|\partial_v L^*(x, v)[v]| \leq C^*(1 + |v|_x^2)$ for all $(x, v) \in TM$. From the first two relations in (3.8) and a theorem of Kranosel'skii we deduce

$$\begin{aligned} \int_0^1 \partial_v L^*(\gamma_k(t), \gamma'_k(t))[\gamma'_k(t)] dt &\rightarrow \int_0^1 \partial_v L^*(\gamma_0(t), \gamma'_0(t))[\gamma'_0(t)] dt, \\ \int_0^1 L^*(\gamma_k(t), \gamma'_k(t)) dt &\rightarrow \int_0^1 L^*(\gamma_0(t), \gamma'_0(t)) dt \end{aligned}$$

and hence $d_k \rightarrow d_0 = c$. Choose $k_0 \in \mathbb{N}$ such that $|d_k| < c + 1$ for all $k \geq k_0$. Then by Claim 3.6 we have a constant $C_5 > 0$ such that

$$|\gamma'_k(t)|_{\gamma_k(t)} \leq C_5 \quad \forall t \in [0, 1] \text{ and } k \geq k_0. \quad (3.9)$$

By the compactness of M and the assumptions (L1)-(L3) for L^* , we may take finite many coordinate charts on M ,

$$\varphi_\alpha : V_\alpha \rightarrow B_{2\rho}^n(0), \quad x \mapsto (x_1^\alpha, \dots, x_n^\alpha), \quad \alpha = 1, \dots, m,$$

and positive constants $C_6 > C_7$, such that $M = \cup_{\alpha=1}^m (\varphi_\alpha)^{-1}(B_\rho^n(0))$ and each

$$L_\alpha^\tau : B_{2\rho}^n(0) \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad (x^\alpha, v^\alpha) \mapsto L^\tau(\varphi_\alpha^{-1}(x^\alpha), d\varphi_\alpha^{-1}(x^\alpha)(v^\alpha))$$

satisfies

$$\begin{aligned} |L_\alpha^\tau(x^\alpha, v^\alpha)| &\leq C_6(1 + |v^\alpha|^2), \\ \left| \frac{\partial L_\alpha^\tau}{\partial x_i^\alpha}(x^\alpha, v^\alpha) \right| &\leq C_6(1 + |v^\alpha|^2), \quad \left| \frac{\partial L_\alpha^\tau}{\partial v_i^\alpha}(x^\alpha, v^\alpha) \right| \leq C_6(1 + |v^\alpha|), \end{aligned} \quad (3.10)$$

$$\left| \frac{\partial^2 L_\alpha^\tau}{\partial x_i^\alpha \partial x_j^\alpha}(x^\alpha, v^\alpha) \right| \leq C_6(1 + |v^\alpha|^2), \quad \left| \frac{\partial^2 L_\alpha^\tau}{\partial x_i^\alpha \partial v_j^\alpha}(x^\alpha, v^\alpha) \right| \leq C_6(1 + |v^\alpha|), \quad (3.11)$$

$$\left| \frac{\partial^2 L_\alpha^\tau}{\partial v_i^\alpha \partial v_j^\alpha}(x^\alpha, v^\alpha) \right| \leq C_6 \quad \text{and} \quad \sum_{ij} \frac{\partial^2 L_\alpha^\tau}{\partial v_i^\alpha \partial v_j^\alpha}(x^\alpha, v^\alpha) u_i u_j \geq C_7 |\mathbf{u}|^2 \quad (3.12)$$

for $\tau \in [0, 1]$, $(x^\alpha, v^\alpha) \in \bar{B}_\rho^n(0) \times \mathbb{R}^n$ and all $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$. Moreover we have also positive constants C_8, C_9 and C_{10} such that

$$L^\tau(x, v) \geq C_8(|v|_q^2 - 1) \quad \forall (x, v) \in TM, \quad (3.13)$$

$$C_9|v^\alpha| \leq |v|_x \leq C_{10}|v^\alpha| \quad \forall v \in T_x M \text{ and } x \in \varphi_\alpha^{-1}(\bar{B}_\rho^n(0)), \quad (3.14)$$

where $v = \sum_{i=1}^n v_i^\alpha \frac{\partial}{\partial x_i^\alpha} \Big|_x$.

For each $k \geq k_0$, set $I_{k,\alpha} := \gamma_k^{-1}(\varphi_\alpha^{-1}(B_\rho^n(0)))$ and $\gamma_{k,\alpha} := \varphi_\alpha \circ \gamma_k : I_{k,\alpha} \rightarrow B_\rho^n(0)$. Then the third condition in (3.8) implies that

$$\begin{aligned} \frac{d}{dt} \partial_{v^\alpha} L_\alpha^{\tau_k}(\gamma_{k,\alpha}(t), \gamma'_{k,\alpha}(t)) &= \partial_{x^\alpha} L_\alpha^{\tau_k}(\gamma_{k,\alpha}(t), \gamma'_{k,\alpha}(t)) \quad \text{or} \\ \gamma''_{k,\alpha}(t) &= (\partial_{v^\alpha} L_\alpha^{\tau_k}(\gamma_{k,\alpha}(t), \gamma'_{k,\alpha}(t)))^{-1} \left[\partial_{x^\alpha} L_\alpha^{\tau_k}(\gamma_{k,\alpha}(t), \gamma'_{k,\alpha}(t)) - \right. \\ &\quad \left. - \partial_{x^\alpha v^\alpha} L_\alpha^{\tau_k}(\gamma_{k,\alpha}(t), \gamma'_{k,\alpha}(t))[\gamma'_{k,\alpha}(t)] \right] \quad \forall t \in I_{k,\alpha}. \end{aligned}$$

It follows that

$$\begin{aligned}
|\gamma''_{k,\alpha}(t)|^2 &= \gamma''_{k,\alpha}(t)(\gamma''_{k,\alpha}(t))^T \\
&= (\partial_{v^\alpha v^\alpha} L_\alpha^{\tau_k})^{-1} \left[\partial_{x^\alpha} L_\alpha^{\tau_k} - \partial_{x^\alpha v^\alpha} L_\alpha^{\tau_k} [\gamma'_{k,\alpha}(t)] \right] (\gamma''_{k,\alpha}(t))^T \\
&\leq \frac{1}{C_7} \left| \partial_{x^\alpha} L_\alpha^{\tau_k} - \partial_{x^\alpha v^\alpha} L_\alpha^{\tau_k} [\gamma'_{k,\alpha}(t)] \right| \cdot |\gamma''_{k,\alpha}(t)|
\end{aligned}$$

because the second inequality of (3.12) implies that $|(\partial_{v^\alpha v^\alpha} L_\alpha^{\tau_k})^{-1} v| \leq |v|/C_7$. Hence

$$\begin{aligned}
|\gamma''_{k,\alpha}(t)| &\leq \frac{1}{C_7} \left| \partial_{x^\alpha} L_\alpha^{\tau_k} - \partial_{x^\alpha v^\alpha} L_\alpha^{\tau_k} [\gamma'_{k,\alpha}(t)] \right| \\
&\leq \frac{C_6}{C_7} (1 + |\gamma'_{k,\alpha}(t)|^2) + \frac{C_6}{C_7} (1 + |\gamma'_{k,\alpha}(t)|) |\gamma'_{k,\alpha}(t)| \\
&\leq 3 \frac{C_6}{C_7} (1 + |\gamma'_{k,\alpha}(t)|^2) \\
&\leq 3 \frac{C_6}{C_7} (1 + |\gamma'_k(t)|^2_{\gamma_k(t)} / C_9^2) \\
&\leq 3 \frac{C_6}{C_7} (1 + C_5^2 / C_9^2)
\end{aligned}$$

by (3.10)-(3.12), (3.14) and (3.9).

Note that $I = \cup_{\alpha=1}^m I_{k,\alpha}$ for each $k \geq k_0$. We may deduce that the sequence (γ_k) is bounded in $C^2([0, 1], M)$. Passing to a subsequence we may assume that (γ_k) converges to $\bar{\gamma} \in C^1([0, 1], M)$ in $C^1([0, 1], M)$. Since the sequence (γ_k) converges to γ_0 in $C^0([0, 1], M)$, $\bar{\gamma} = \gamma_0$. That is, (γ_k) converges to γ_0 in $C^1([0, 1], M)$. It follows that $\min_{t \in [0, 1]} L(\gamma_k(t), \gamma'_k(t)) > \frac{2c}{3}$ for sufficiently large k . This and (3.7)-(3.8) lead to $d\mathcal{L}(\gamma_k) = 0$ for sufficiently large k , which contradicts to the assumption that $\gamma_0 \in \Lambda_N(M)$ is an isolated nonconstant critical point of \mathcal{L} on $\Lambda_N(M)$. \square

Using Theorem 3.3 and Claims 3.4, 3.5 and 3.7 we immediately get the following key result for the proof of Theorem 1.2(ii).

Theorem 3.8 *Let $\gamma_0 \in \Lambda_N(M)$ be an isolated nonconstant critical point of \mathcal{L} on $\Lambda_N(M)$ (and hence a C^k nonconstant F -geodesics with constant speed $F(\gamma_0(t), \dot{\gamma}_0(t)) \equiv \sqrt{c} > 0$). Let $L^* : TM \rightarrow \mathbb{R}$ be given by Corollary 2.3. Then γ_0 is a uniformly isolated critical point of the family of functionals $\{\mathcal{L}^\tau = (1 - \tau)\mathcal{L} + \tau\mathcal{L}^* \mid 0 \leq \tau \leq 1\}$ too, and $C_*(\mathcal{L}^\tau, \gamma_0; \mathbb{K})$ is independent of $\tau \in [0, 1]$.*

Next we consider the case $N = \Delta_M$ and assume $k \geq 4$.

Claim 3.9 *Let $S^1 \cdot \gamma_0 \subset \Lambda M = W^{1,2}(S^1, M)$ be an isolated nonconstant critical orbit of \mathcal{L} on ΛM (with $F(\gamma_0(t), \dot{\gamma}_0(t)) \equiv \sqrt{c} > 0$). Let $L^* : TM \rightarrow \mathbb{R}$ be given by Corollary 2.3, and let $L^\tau(x, v) = (1 - \tau)L(x, v) + \tau L^*(x, v)$ for $\tau \in [0, 1]$. Then there exists a neighborhood of $S^1 \cdot \gamma_0$ in ΛM , $\mathcal{U}(S^1 \cdot \gamma_0)$, such that the critical set of each \mathcal{L}^τ in $\mathcal{U}(S^1 \cdot \gamma_0)$ is the orbit $S^1 \cdot \gamma_0$.*

Proof. From the beginning of the proof of Claim 3.7 one easily sees that $S^1 \cdot \gamma_0$ is a critical orbit of each \mathcal{L}^τ . As in the proof of Claim 3.7, by a contradiction, suppose that there exist sequences $\{\gamma_k\} \subset \Lambda M$, $\{\tau_k\} \subset [0, 1]$ and $[s] \in S^1$ such that

$$\tau_k \rightarrow \tau_0, \quad \gamma_k \rightarrow [s] \cdot \gamma_0, \quad d\mathcal{L}^{\tau_k}(\gamma_k) = 0 \text{ and } \gamma_k \notin S^1 \cdot \gamma_0 \quad \forall k.$$

Then it was shown at the end of the proof of Claim 3.7 that $d\mathcal{L}(\gamma_k) = 0$ for sufficiently large k . By the assumption that $S^1 \cdot \gamma_0 \subset \Lambda M$ is an isolated nonconstant critical orbit of \mathcal{L} on ΛM , we obtain $\gamma_k = [s_k] \cdot \gamma_0$ for some $[s_k] \in S^1$ and each sufficiently large k . This contradiction gives our claim. \square

Theorem 1.5(ii) follows from Claim 3.9 and the following result.

Theorem 3.10 *Under the assumptions of Claim 3.9, $C_*(\mathcal{L}^\tau, S^1 \cdot \gamma_0; \mathbb{K})$ is independent of $\tau \in [0, 1]$.*

Proof. Since $L^\tau = (1 - \tau)L + \tau L^*$ and $L([s] \cdot \gamma_0(t), ([s] \cdot \gamma_0)'(t)) = c$ for all $t \in \mathbb{R}$ and $[s] \in S^1$, we have

$$\mathcal{L}^\tau([s] \cdot \gamma_0) = c \quad \forall \tau \in [0, 1], [s] \in S^1. \quad (3.15)$$

Claim 3.9 yields a neighborhood of $S^1 \cdot \gamma_0$ in ΛM , $\mathcal{U}(S^1 \cdot \gamma_0)$, such that

$$\mathcal{U}(S^1 \cdot \gamma_0) \cap K(\mathcal{L}^\tau) = S^1 \cdot \gamma_0 \quad \forall \tau \in [0, 1], \quad (3.16)$$

where $K(\mathcal{L}^\tau)$ denotes the critical set of \mathcal{L}^τ .

Denote by $\mathcal{O} = S^1 \cdot \gamma_0$. Since γ_0 is nonconstant, it has minimal period $1/m$ for some $m \in \mathbb{N}$, and \mathcal{O} is a 1-dimensional C^3 -submanifold diffeomorphic to the circle ([25, page 499]). For every $s \in [0, 1/m]$ the tangent space $T_{[s] \cdot \gamma_0}(S^1 \cdot \gamma_0)$ is $\mathbb{R}([s] \cdot \gamma_0)'$, and the fiber $N\mathcal{O}_{[s] \cdot \gamma_0}$ at $[s] \cdot \gamma_0$ of the normal bundle $N\mathcal{O}$ of \mathcal{O} is a subspace of codimension 1 which is orthogonal to $([s] \cdot \gamma_0)'$ in $T_{[s] \cdot \gamma_0}\Lambda M$. For sufficiently small $\delta > 0$, as in [32, §5] we have a C^2 -diffeomorphism from an open neighborhood of the zero section of $N\mathcal{O}$,

$$N\mathcal{O}(\delta) := \{(y, v) \in N\mathcal{O} \mid y \in \mathcal{O}, \|v\|_1 < \delta\},$$

to an open neighborhood $U(\mathcal{O})$ of \mathcal{O} in ΛM ,

$$\Psi = \text{EXP}|_{N\mathcal{O}(\delta)} : N\mathcal{O}(\delta) \rightarrow U(\mathcal{O}). \quad (3.17)$$

By shrinking $\delta > 0$ we can require that $U(\mathcal{O}) \subset \mathcal{U}(S^1 \cdot \gamma_0)$ (hence $U(\mathcal{O})$ contains no other critical orbit besides \mathcal{O}), and that $\Psi(\{y\} \times N\mathcal{O}(\delta)_y)$ and \mathcal{O} have a unique intersection point y (after identifying \mathcal{O} with the zero section of $N\mathcal{O}$), where $N\mathcal{O}(\delta)_y := N\mathcal{O}(\delta) \cap N\mathcal{O}_y$ and $\mathcal{U}(S^1 \cdot \gamma_0)$ is as in Claim 3.9. Clearly,

$$\Psi(y, 0) = y \quad \text{and} \quad \Psi([s] \cdot y, [s] \cdot v) = [s] \cdot \Psi_\tau(y, v)$$

for any $(y, v) \in N\mathcal{O}(\delta)$ and $[s] \in S^1$. It follows that $U(\mathcal{O})$ is an S^1 -invariant neighborhood of \mathcal{O} , and that Ψ is S^1 -equivariant. For \mathcal{L}^τ in (3.15), define

$$\mathcal{F}^\tau : N\mathcal{O}(\delta) \rightarrow \mathbb{R}, \quad (y, v) \mapsto \mathcal{L}^\tau \circ \Psi(y, v). \quad (3.18)$$

It is C^{2-0} , and satisfies the (PS) condition and

$$\mathcal{F}^\tau([s] \cdot y, [s] \cdot v) = \mathcal{F}^\tau(y, v) \quad \forall (y, v) \in N\mathcal{O}(\delta) \text{ and } [s] \in S^1.$$

Since $C_*(\mathcal{L}^\tau, \mathcal{O}; \mathbb{K}) = C_*(\mathcal{F}^\tau, \mathcal{O}; \mathbb{K})$ for any $\tau \in [0, 1]$, we only need to prove that

$$C_*(\mathcal{F}^\tau, \mathcal{O}; \mathbb{K}) \quad \text{is independent of } t. \quad (3.19)$$

For $\sigma, \tau \in [0, 1]$ and $\gamma \in \Lambda M$, we have (3.4) and

$$\nabla \mathcal{L}^\sigma(\gamma) - \nabla \mathcal{L}^\tau(\gamma) = (\tau - \sigma)(\nabla \mathcal{L}(\gamma) - \nabla \mathcal{L}^*(\gamma)).$$

From them it follows that there exists a constant $C_{11} > 0$ such that

$$\begin{aligned} & \sup\{|\mathcal{F}^\sigma(y, v) - \mathcal{F}^\tau(y, v)| + |\nabla \mathcal{F}^\sigma(y, v) - \nabla \mathcal{F}^\tau(y, v)| : (y, v) \in N(\mathcal{O})(\delta)\} \\ & \leq C_{11}|\sigma - \tau| \end{aligned} \quad (3.20)$$

after shrinking $\delta > 0$ (if necessary) because \mathcal{O} is compact. Using these (3.19) easily follows from Chang and Ghoussoub [18, Th.III.4]. We may also prove (3.19) as follows. For each fixed a $\tau \in [0, 1]$, following [46, Th.2.3] we may construct a Gromoll-Meyer pair of \mathcal{O} as a critical submanifold of \mathcal{F}^τ on $N(\mathcal{O})(\delta)$ with respect to $-\nabla \mathcal{F}^\tau$, $(W(\mathcal{O}), W(\mathcal{O})^-)$, such that

$$(W(\mathcal{O})_y, W(\mathcal{O})_y^-) := (W(\mathcal{O}) \cap N(\mathcal{O})(\delta)_y, W(\mathcal{O})^- \cap N(\mathcal{O})(\delta)_y) \quad (3.21)$$

is a Gromoll-Meyer pair of $\mathcal{F}^\tau|_{N(\mathcal{O})(\delta)_y}$ at its isolated critical point $0 = (y, 0)$ satisfying

$$(W(\mathcal{O})_{[s] \cdot y}, W(\mathcal{O})_{[s] \cdot y}^-) = ([s] \cdot W(\mathcal{O})_y, [s] \cdot W(\mathcal{O})_y^-)$$

for any $[s] \in S^1$ and $y \in \mathcal{O}$.

Slightly modifying the proof of Lemma 5.2 on the page 52 of [16] we may show that $(W(\mathcal{O}), W(\mathcal{O})^-)$ is also a Gromoll-Meyer pair of \mathcal{O} as a critical submanifold of \mathcal{F}^σ on $N(\mathcal{O})(\delta)$ with respect to certain pseudo-gradient vector field of \mathcal{F}^σ if $\sigma \in [0, 1]$ is sufficiently close to τ because of (3.20). So we may get an open neighborhood J_τ of $\tau \in [0, 1]$ in $I = [0, 1]$ such that

$$C_*(\mathcal{F}^\tau, \mathcal{O}; \mathbb{K}) = C_*(\mathcal{F}^\sigma, \mathcal{O}; \mathbb{K}) \quad \forall \sigma \in J_\tau.$$

Then (3.19) follows from this and the compactness of $[0, 1]$. \square

4 Proofs of Theorems 1.2, 1.3

Step 1. *Prove the corresponding versions of Theorems 1.2, 1.3 under a new chart around γ_0 .*

For conveniences of computations we need to consider a coordinate chart around γ_0 different from that of (1.4). Recall that M_0 (resp. M_1) is totally geodesic near $\gamma_0(0)$ (resp. $\gamma_0(1)$) with respect to the chosen Riemannian metric h on M . Let

$I \ni t \rightarrow (e_1(t), \dots, e_n(t))$ be a parallel orthonormal frame along γ_0 with respect to the metric h . For some small open ball $B_{2\rho}^n(0) \subset \mathbb{R}^n$ we get a smooth map $\phi : I \times B_{2\rho}^n(0) \rightarrow M$ given by

$$\phi(t, v) = \exp_{\gamma_0(t)} \left(\sum_{i=1}^n v_i e_i(t) \right). \quad (4.1)$$

Since there exist linear subspaces $V_i \subset \mathbb{R}^n$, $i = 0, 1$, such that

$$v \in V_i \iff \sum_{k=1}^n v_k e_k(i) \in T_{\gamma_0(i)} M_i, \quad i = 0, 1,$$

by shrinking $\rho > 0$ (if necessary) we get

$$v \in V_i \cap B_{2\rho}^n(0) \iff \phi(i, v) \in M_i, \quad i = 0, 1.$$

Set $V := V_0 \times V_1$ and

$$\begin{aligned} H_V &= W_V^{1,2}(I, \mathbb{R}^n) := \{\xi \in W^{1,2}(I, \mathbb{R}^n) \mid (\xi(0), \xi(1)) \in V\}, \\ X_V &= C_V^1(I, \mathbb{R}^n) := \{\xi \in C^1(I, \mathbb{R}^n) \mid (\xi(0), \xi(1)) \in V\}. \end{aligned}$$

Let $\mathbf{B}_{2\rho}(H_V) := \{\xi \in H_V \mid \|\xi\|_1 < 2\rho\}$. Then the map

$$\Phi : \mathbf{B}_{2\rho}(H_V) \rightarrow \Lambda_N(M) \quad (4.2)$$

defined by $\Phi(\xi)(t) = \phi(t, \xi(t))$, gives a C^k coordinate chart around γ_0 on $\Lambda_N(M)$ with $\text{Im}(\Phi) \subset \mathcal{O}(\gamma_0)$ (because F is only C^k on $TM \setminus \{0\}$ ($k \geq 2$), and hence γ_0 is C^k). Define $\tilde{F} : I \times B_{2\rho}^n(0) \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\tilde{F}(t, x, v) = F(\phi(t, x), D\phi(t, x)[(1, v)]).$$

Then $\tilde{F}(t, 0, 0) = F(\phi(t, 0), D_t\phi(t, x)[1]) = F(\gamma_0(t), \dot{\gamma}_0(t)) \equiv \sqrt{c}$. Moreover the C^{2-0} function $\tilde{L} := \tilde{F}^2$ satisfies $\tilde{L}(t, x, v) = L(\phi(t, x), D\phi(t, x)[(1, v)])$, and is C^k in $(I \times B_{2\rho}^n(0) \times \mathbb{R}^n) \setminus \mathcal{Z}$, where

$$\mathcal{Z} := \{(t, x, v) \in I \times B_{2\rho}^n(0) \times \mathbb{R}^n \mid \partial_x \phi(t, x)[v] = -\partial_t \phi(t, x)\},$$

a relative closed subset in $I \times B_{2\rho}^n(0) \times \mathbb{R}^n$. Since γ_0 is regular, i.e., $\partial_t \phi(t, 0) = \dot{\gamma}_0(t) \neq 0$ at each $t \in I$, and $\partial_x \phi(t, x)$ is injective, we deduce that $(t, 0, 0) \notin \mathcal{Z} \forall t \in I$. It follows that $I \times B_{2r}^n(0) \times B_{2r}^n(0) \subset I \times B_{2\rho}^n(0) \times \mathbb{R}^n \setminus \mathcal{Z}$ for some $0 < r < \rho$. We also require $r > 0$ so small that

$$\tilde{L}(t, x, v) \geq \frac{2c}{3} \quad \forall (t, x, v) \in I \times B_r^n(0) \times B_r^n(0). \quad (4.3)$$

Denote by $\mathbf{B}_{2r}(X_V) = \{\xi \in X_V \mid \|\xi\|_{C^1} < 2r\}$. Then $\mathbf{B}_{2r}(X_V) \subset \mathbf{B}_{2r}(H_V)$. Define the action functional

$$\tilde{\mathcal{L}} : \mathbf{B}_{2r}(H_V) \rightarrow \mathbb{R}, \quad \xi \mapsto \tilde{\mathcal{L}}(\xi) = \int_0^1 \tilde{L}(t, \xi(t), \dot{\xi}(t)) dt, \quad (4.4)$$

that is, $\tilde{\mathcal{L}} = \mathcal{L} \circ \Phi$. It is C^{2-0} , has $0 \in \mathbf{B}_{2r}(H_V)$ as a unique critical point.

We conclude that the boundary condition (1.3) becomes

$$\partial_v \tilde{L}(i, 0, 0)[v] = 0 \quad \forall v \in V_i, \quad i = 0, 1. \quad (4.5)$$

In fact, for any $X \in T_{\gamma(0)}M_0$ there exists a unique $v = (v_1, \dots, v_n) \in V_0$ such that $X = \sum_{k=1}^n v_k e_k(0) = \partial_v \phi(0, 0)[v]$. Since $\gamma_0(0) = \phi(0, 0)$ and $\dot{\gamma}_0(0) = \partial_t \phi(0, 0)$ we get

$$\begin{aligned} 0 &= g^F(\gamma_0(0), \dot{\gamma}_0(0))[X, \dot{\gamma}_0(0)] \\ &= \left. \frac{d^2}{dsd\tau} \right|_{s=0, \tau=0} L(\gamma_0(0), \dot{\gamma}_0(0) + sX + \tau\dot{\gamma}_0(0)) \\ &= \left. \frac{d^2}{dsd\tau} \right|_{s=0, \tau=0} L(\phi(0, 0), \partial_t \phi(0, 0) + s\partial_v \phi(0, 0)[v] + \tau\partial_t \phi(0, 0)) \\ &= \left. \frac{d^2}{dsd\tau} \right|_{s=0, \tau=0} \left[(1 + \tau)^2 L \left(\phi(0, 0), \partial_t \phi(0, 0) + \frac{s}{1 + \tau} \partial_v \phi(0, 0)[v] \right) \right] \\ &= \left. \frac{d^2}{dsd\tau} \right|_{s=0, \tau=0} \left[(1 + \tau)^2 L \left(\phi(0, 0), D\phi(0, 0)[1, \frac{s}{1 + \tau}v] \right) \right] \\ &= \left. \frac{d^2}{dsd\tau} \right|_{s=0, \tau=0} \left[(1 + \tau)^2 \tilde{L} \left(0, 0, \frac{s}{1 + \tau}v \right) \right] \\ &= \left. \frac{d}{ds} \right|_{s=0} \left[2\tilde{L}(0, 0, sv) - s\partial_v \tilde{L}(0, 0, sv)[v] \right] \\ &= \partial_v \tilde{L}(0, 0, 0)[v]. \end{aligned}$$

Similarly, we may prove that $g^F(\gamma_0(1), \dot{\gamma}_0(1))[X, \dot{\gamma}_0(1)] = 0 \quad \forall X \in T_{\gamma_0(1)}M_1$ if and only if $\partial_v \tilde{L}(1, 0, 0)[v] = 0 \quad \forall v \in V_1$.

Let \tilde{A} denote the restriction of the gradient $\nabla \tilde{\mathcal{L}}$ to $\mathbf{B}_{2r}(X_V)$. Then $\tilde{A}(\mathbf{B}_{2r}(X_V)) \subset \mathbf{B}_{2r}(X_V)$ and $\tilde{A} : \mathbf{B}_{2r}(X_V) \rightarrow \mathbf{B}_{2r}(X_V)$ is C^1 . (This implies that the restriction of $\tilde{\mathcal{L}}$ to $\mathbf{B}_{2r}(X_V)$ is C^2 .) The continuous symmetric bilinear form $d^2 \tilde{\mathcal{L}}|_{\mathbf{B}_{2r}(X_V)}(0)$ on X_V can be extended into such a form on H_V whose associated self-adjoint operator is Fredholm, and has finite dimensional negative definite and null spaces H_V^- and H_V^0 , which are actually contained in X_V . Let H_V^+ be the corresponding positive definite space. Then the orthogonal decomposition $H_V = H_V^- \oplus H_V^0 \oplus H_V^+$ induces a (topological) direct sum decomposition $X_V = X_V^- \oplus X_V^0 \oplus X_V^+$, where as sets $X_V^0 = H_V^0 = \text{Ker}(D\tilde{A}(0))$, $X_V^- = H_V^-$ and $X_V^+ = X_V \cap H_V^+$. Note that H_V and X_V induce equivalent norms on $H_V^0 = X_V^0$. By the implicit function theorem we get a $\tau \in (0, r]$ and a C^1 -map $\tilde{h} : \mathbf{B}_\tau(H_V^0) \rightarrow X_V^- \oplus X_V^+$ with $\tilde{h}(0) = 0$ such that for each $\xi \in \mathbf{B}_\tau(H_V^0)$,

$$\xi + \tilde{h}(\xi) \in \mathbf{B}_{2r}(X_V^0) \quad \text{and} \quad (I - P_V^0)\tilde{A}(\xi + \tilde{h}(\xi)) = 0, \quad (4.6)$$

where $P_V^0 : H_V \rightarrow H_V^0$ is the orthogonal projection. It is not hard to prove that the Morse index $m^-(\gamma_0)$ and nullity $m^0(\gamma_0)$ of γ_0 are equal to $\dim X_V^-$ and $\dim X_V^0$, respectively.

Let $\tilde{L}^*(t, x, v) = L^*(\phi(t, x), D\phi(t, x)[(1, v)])$. Since $L^*(x, v) = L(x, v)$ if $L(x, v) \geq \frac{2c}{3}$, $L^*(\gamma_0(t), \dot{\gamma}_0(t)) = c \quad \forall t$, and

$$\tilde{L}^*(t, x, v) = \tilde{L}(t, x, v) \quad \text{if} \quad \tilde{L}(t, x, v) \geq \frac{2c}{3}. \quad (4.7)$$

In particular, $\tilde{L}(t, 0, 0) = c$ implies that $\tilde{L}^*(t, 0, 0) \equiv c \forall t$. Define the action functional

$$\tilde{\mathcal{L}}^* : \mathbf{B}_{2r}(H_V) \rightarrow \mathbb{R}, \xi \mapsto \int_0^1 \tilde{L}^*(t, \xi(t), \dot{\xi}(t)) dt,$$

which is C^{2-0} , has $0 \in \mathbf{B}_{2r}(H_V)$ as a unique critical point.

We firstly check that $\tilde{\mathcal{L}}^*$ satisfies the conditions of Theorem A.1. This can be obtained by almost repeating the arguments in [32, §3]. For the sake of completeness we also give them. It is easily computed (cf. [32, §3]) that

$$d\tilde{\mathcal{L}}^*(\tilde{\gamma})(\xi) = \int_0^1 \left(D_q \tilde{L}^*(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) \cdot \xi(t) + D_v \tilde{L}^*(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) \cdot \dot{\xi}(t) \right) dt$$

for any $\tilde{\gamma} \in \mathbf{B}_{2r}(H_V)$ and $\xi \in H_V$. Write $U = \mathbf{B}_{2r}(H_V)$ and

$$U_X := \mathbf{B}_{2r}(H_V) \cap X_V \text{ as an open subset of } X_V.$$

Then the restriction of $\tilde{\mathcal{L}}^*$ to U_X , denoted by $\tilde{\mathcal{L}}_X^*$, is at least C^2 and

$$\begin{aligned} d^2 \tilde{\mathcal{L}}_X^*(\tilde{\gamma})(\xi, \eta) = \int_0^1 & \left(D_{vv} \tilde{L}^*(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) (\dot{\xi}(t), \dot{\eta}(t)) \right. \\ & + D_{qv} \tilde{L}^*(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) (\xi(t), \dot{\eta}(t)) \\ & + D_{vq} \tilde{L}^*(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) (\dot{\xi}(t), \eta(t)) \\ & \left. + D_{qq} \tilde{L}^*(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) (\xi(t), \eta(t)) \right) dt \end{aligned} \quad (4.8)$$

for any $\tilde{\gamma} \in U_X$, $\xi, \eta \in X_V$. Let us compute the gradient $\nabla \tilde{\mathcal{L}}^*(\tilde{\gamma})$. Define

$$G(\tilde{\gamma})(t) := \int_0^t \left[D_v \tilde{L}^*(s, \tilde{\gamma}(s), \dot{\tilde{\gamma}}(s)) - c_0 \right] ds \quad (4.9)$$

for $t \in [0, 1]$, where $c_0 = \int_0^1 D_v \tilde{L}^*(s, \tilde{\gamma}(s), \dot{\tilde{\gamma}}(s)) ds$. Then $G(\tilde{\gamma})(0) = 0 = G(\tilde{\gamma})(1)$, and hence $G(\tilde{\gamma}) \in H_0^1([0, 1], \mathbb{R}^n) \subset H_V^1([0, 1], \mathbb{R}^n)$. Moreover

$$\begin{aligned} & \int_0^1 \left(D_q \tilde{L}^*(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) \cdot \xi(t) + D_v \tilde{L}^*(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) \cdot \dot{\xi}(t) \right) dt \\ & = (G(\tilde{\gamma}), \xi)_{W^{1,2}} + c_0 \int_0^1 \dot{\xi}(t) dt + \int_0^1 \left(D_q \tilde{L}^*(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) - G(\tilde{\gamma})(t) \right) \cdot \xi(t) dt. \end{aligned}$$

By the Riesz theorem one may get an unique $F(\tilde{\gamma}) \in W_V^{1,2}(I, M)$ such that

$$c_0 \int_0^1 \dot{\xi}(t) dt + \int_0^1 \left(D_q \tilde{L}^*(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) - G(\tilde{\gamma})(t) \right) \cdot \xi(t) dt = (F(\tilde{\gamma}), \xi)_{W^{1,2}} \quad (4.10)$$

for any $\xi \in W_V^{1,2}(I, \mathbb{R}^n)$. Hence $d\tilde{\mathcal{L}}^*(\tilde{\gamma})(\xi) = (G(\tilde{\gamma}), \xi)_{W^{1,2}} + (F(\tilde{\gamma}), \xi)_{W^{1,2}}$ and thus

$$\nabla \tilde{\mathcal{L}}^*(\tilde{\gamma}) = G(\tilde{\gamma}) + F(\tilde{\gamma}). \quad (4.11)$$

Since

$$(F(\tilde{\gamma}), \xi)_{W^{1,2}} = \int_0^1 \left(F(\tilde{\gamma})(t) \cdot \xi(t) + \frac{d}{dt} F(\tilde{\gamma})(t) \cdot \dot{\xi}(t) \right) dt,$$

(4.10) becomes

$$\begin{aligned} & \int_0^1 \left(D_q \tilde{L}^* (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) - G(\tilde{\gamma})(t) - F(\tilde{\gamma})(t) \right) \cdot \xi(t) dt \\ &= \int_0^1 \left(\frac{d}{dt} F(\tilde{\gamma})(t) - c_0 \right) \cdot \dot{\xi}(t) dt \quad \forall \xi \in W_V^{1,2}(I, \mathbb{R}^n). \end{aligned} \quad (4.12)$$

Lemma 4.1 For $f \in L^1(I, \mathbb{R}^n)$ the equation

$$\ddot{x}(t) - x(t) = f(t)$$

has the general solution of the following form

$$x(t) = e^t \int_0^t \left[e^{-2s} \int_0^s e^\tau f(\tau) d\tau \right] ds + c_1 e^t + c_2 e^{-t},$$

where $c_i \in \mathbb{R}^n$, $i = 1, 2$, are constant vectors.

Setting $y(t) = \dot{x}(t) - x(t)$, this lemma can easily be proved by the standard methods. Let constant vectors $c_1, c_2 \in \mathbb{R}^n$ be such that the function

$$z(t) := e^t \int_0^t \left[e^{-2s} \int_0^s e^\tau f(\tau) d\tau \right] ds + c_1 e^t + c_2 e^{-t} \quad (4.13)$$

satisfies $z(0) = F(\tilde{\gamma})(0)$ and $z(1) = F(\tilde{\gamma})(1) - c_0$ with

$$\begin{aligned} f(t) &= -D_q \tilde{L}^* (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) + G(\tilde{\gamma})(t) + c_0 t \\ &= -D_q \tilde{L}^* (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) + \int_0^t D_q \tilde{L}^* (s, \tilde{\gamma}(s), \dot{\tilde{\gamma}}(s)) ds. \end{aligned} \quad (4.14)$$

(We identify each element of $W^{1,2}(I, \mathbb{R}^n)$ with its unique continuous representation as usual). Then for any $\xi \in C^1(I, \mathbb{R}^n)$ with $\xi(0) = \xi(1) = 0$ it holds that

$$\begin{aligned} \int_0^1 \dot{z}(t) \cdot \dot{\xi}(t) dt &= z(t) \cdot \dot{\xi}(t) \Big|_{t=0}^{t=1} - \int_0^1 \ddot{z}(t) \cdot \xi(t) dt \\ &= - \int_0^1 (z(t) + f(t)) \cdot \xi(t) dt \\ &= \int_0^1 (D_q \tilde{L}^* (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) - G(\tilde{\gamma})(t) - c_0 t - z(t)) \cdot \xi(t) dt. \end{aligned}$$

From this and (4.12) it follows that

$$\int_0^1 (F(\tilde{\gamma})(t) - c_0 t - z(t)) \cdot \xi(t) dt = - \int_0^1 \left(\frac{d}{dt} F(\tilde{\gamma})(t) - c_0 - \dot{z}(t) \right) \cdot \dot{\xi}(t) dt$$

for any $\xi \in C^1(I, \mathbb{R}^n)$ with $\xi(0) = \xi(1) = 0$. Since $F(\tilde{\gamma})(t) - c_0 t - z(t)$ is equal to the zero at $t = 0, 1$, by Theorem 8.7 in [9] there exists a sequence $(u_k) \in C_0^\infty(\mathbb{R})$ such that $(u_k|_I)_k$ converges to $F(\tilde{\gamma})(t) - c_0 t - z(t)$ in $W^{1,2}(I, \mathbb{R}^n)$ (and hence in $C(I, \mathbb{R}^n)$). In particular we have $u_k(i) \rightarrow 0$ because $F(\tilde{\gamma})(i) - c_0 \cdot i - z(i) = 0$, $i = 0, 1$. Define $v_k : I \rightarrow \mathbb{R}^n$ by $v_k(t) = u_k(t) - u_k(0) - t(u_k(0) - u_k(1))$ for each $k \in \mathbb{N}$.

Then $v_k \in C^\infty(I, \mathbb{R}^n)$ and $v_k(0) = v_k(1) = 0$ for each k , and $(v_k)_k$ converges to $F(\tilde{\gamma})(t) - c_0 t - z(t)$ in $W^{1,2}(I, \mathbb{R}^n)$. Let $k \rightarrow \infty$ in

$$\int_0^1 (F(\tilde{\gamma})(t) - c_0 t - z(t)) \cdot v_k(t) dt = - \int_0^1 \left(\frac{d}{dt} F(\tilde{\gamma})(t) - c_0 - \dot{z}(t) \right) \cdot \dot{v}_k(t) dt,$$

we obtain

$$\int_0^1 |F(\tilde{\gamma})(t) - c_0 t - z(t)|^2 dt = - \int_0^1 \left| \frac{d}{dt} F(\tilde{\gamma})(t) - c_0 - \dot{z}(t) \right|^2 dt$$

and therefore $F(\tilde{\gamma})(t) = c_0 t + z(t) \forall t \in I$ since both $F(\tilde{\gamma})$ and z are continuous on I . By (4.13), (4.9) and (4.11) we arrive at

$$\begin{aligned} F(\tilde{\gamma})(t) &= e^t \int_0^t \left[e^{-2s} \int_0^s e^\tau f(\tau) d\tau \right] ds + c_1 e^t + c_2 e^{-t} + c_0 t, \\ \nabla \tilde{\mathcal{L}}^*(\tilde{\gamma})(t) &= e^t \int_0^t \left[e^{-2s} \int_0^s e^\tau f(\tau) d\tau \right] ds + c_1 e^t + c_2 e^{-t} \\ &\quad + \int_0^t D_v \tilde{L}^*(s, \tilde{\gamma}(s), \dot{\tilde{\gamma}}(s)) ds, \end{aligned} \quad (4.15)$$

where $c_1, c_2 \in \mathbb{R}^n$ are suitable constant vectors and $f(t)$ is given by (4.14). By (4.15) the function $\nabla \tilde{\mathcal{L}}^*(\tilde{\gamma})(t)$ is differentiable almost everywhere, and for a.e. $t \in I$,

$$\begin{aligned} \frac{d}{dt} \nabla \tilde{\mathcal{L}}^*(\tilde{\gamma})(t) &= e^t \int_0^t \left[e^{-2s} \int_0^s e^\tau f(\tau) d\tau \right] ds + e^{-t} \int_0^t e^\tau f(\tau) d\tau \\ &\quad + c_1 e^t - c_2 e^{-t} + D_v \tilde{L}^*(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)). \end{aligned} \quad (4.16)$$

Let \tilde{A}^* denote the restriction of the gradient $\nabla \tilde{\mathcal{L}}^*$ to U_X . Clearly, (4.15) and (4.16) imply that $\tilde{A}^*(\tilde{\gamma}) \in X_V$ for $\tilde{\gamma} \in U_X$, and that $U_X \ni \tilde{\gamma} \mapsto \tilde{A}^*(\tilde{\gamma}) \in X_V$ is continuous. Furthermore, as in the proof of [32, Lemma 3.2] we have also

Lemma 4.2 *The map $\tilde{A}^* : U_X \rightarrow X_V$ is continuously differentiable.*

From (4.8) it easily follows that

(i) for any $\tilde{\gamma} \in U_X$ there exists a constant $C(\tilde{\gamma})$ such that

$$|d^2 \tilde{\mathcal{L}}_X^*(\tilde{\gamma})(\xi, \eta)| \leq C(\tilde{\gamma}) \|\xi\|_{W^{1,2}} \cdot \|\eta\|_{W^{1,2}} \quad \forall \xi, \eta \in X_V;$$

(ii) $\forall \varepsilon > 0, \exists \delta_0 > 0$, such that for any $\tilde{\gamma}_1, \tilde{\gamma}_2 \in U_X$ with $\|\tilde{\gamma}_1 - \tilde{\gamma}_2\|_{C^1} < \delta_0$,

$$|d^2 \tilde{\mathcal{L}}_X^*(\tilde{\gamma}_1)(\xi, \eta) - d^2 \tilde{\mathcal{L}}_X^*(\tilde{\gamma}_2)(\xi, \eta)| \leq \varepsilon \|\xi\|_{W^{1,2}} \cdot \|\eta\|_{W^{1,2}} \quad \forall \xi, \eta \in X_V.$$

(i) shows that the right side of (4.8) is also a bounded symmetric bilinear form on H_V . As in [32, §3] we have a map $\tilde{B}^* : U_X \rightarrow L_s(H_V)$, which is uniformly continuous, such that

$$(D\tilde{A}^*(\tilde{\gamma})\xi, \eta)_{W^{1,2}} = d^2 \tilde{\mathcal{L}}_X^*(\tilde{\gamma})(\xi, \eta) = (\tilde{B}^*(\tilde{\gamma})\xi, \eta)_{W^{1,2}} \quad (4.17)$$

for any $\tilde{\gamma} \in U_X$ and $\xi, \eta \in X_V$. Namely (A.2) is satisfied. Almost repeating the arguments in [32, §3] we may check that the map \tilde{B}^* satisfy the conditions **(B1)** and **(B2)** in Appendix A. Summarizing the above arguments, $(\tilde{\mathcal{L}}, \tilde{A}^*, \tilde{B}^*)$ satisfies the conditions of Theorem A.1 (resp. Theorem A.5) around the critical point $0 \in H_V$ (resp. $0 \in X_V$). By (4.3) and (4.7) for any $\tilde{\gamma} \in \mathbf{B}_r(X_V)$ we have

$$\tilde{L}^*(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) = \tilde{L}(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) \quad \forall t \in I. \quad (4.18)$$

This implies that

$$\tilde{A}^*(\tilde{\gamma}) = \tilde{A}(\tilde{\gamma}) \quad \forall \tilde{\gamma} \in \mathbf{B}_r(X_V), \quad \text{and} \quad \tilde{B}^*(0) = \tilde{B}(0), \quad (4.19)$$

where the map $\tilde{B} : \mathbf{B}_{2r}(X_V) \rightarrow L_s(H_V)$ is determined by the equation

$$d^2 \tilde{\mathcal{L}}_X(\tilde{\gamma})(\xi, \eta) = (\tilde{B}(\tilde{\gamma})\xi, \eta)_{W^{1,2}} \quad \forall \tilde{\gamma} \in \mathbf{B}_{2r}(X_V), \quad \xi, \eta \in X_V.$$

By shrinking δ in (4.6) so small that

$$\xi + \tilde{h}(\xi) \in \mathbf{B}_r(X_V) \quad \forall \xi \in \mathbf{B}_\tau(H_V^0), \quad (4.20)$$

it follows from this and (4.19) that the C^1 -map \tilde{h} in (4.6) satisfies

$$(I - P_V^0)\tilde{A}^*(\xi + \tilde{h}(\xi)) = 0 \quad \forall \xi \in \mathbf{B}_\tau(H_V^0).$$

Let $\tilde{\mathcal{L}}^{*\circ}, \tilde{\mathcal{L}}^\circ : \mathbf{B}_\tau(H_V^0) \rightarrow \mathbb{R}$ be defined by

$$\tilde{\mathcal{L}}^{*\circ}(\xi) = \tilde{\mathcal{L}}^*(\xi + \tilde{h}(\xi)) \quad \text{and} \quad \tilde{\mathcal{L}}^\circ(\xi) = \tilde{\mathcal{L}}(\xi + \tilde{h}(\xi)). \quad (4.21)$$

Then (4.20) and (4.18) lead to

$$\tilde{\mathcal{L}}^{*\circ}(\xi) = \tilde{\mathcal{L}}^*(\xi + \tilde{h}(\xi)) = \tilde{\mathcal{L}}(\xi + \tilde{h}(\xi)) = \tilde{\mathcal{L}}^\circ(\xi) \quad (4.22)$$

for all $\xi \in \mathbf{B}_\tau(H_V^0)$. By Theorem A.1 we obtain the following splitting theorem.

Theorem 4.3 *Under the notations above, there exists a ball $\mathbf{B}_\eta(H_V) \subset \mathbf{B}_\tau(H_V)$, an origin-preserving local homeomorphism $\tilde{\psi}$ from $\mathbf{B}_\eta(H_V)$ to an open neighborhood of $0 \in H_V$ such that*

$$\tilde{\mathcal{L}}^* \circ \tilde{\psi}(\xi) = \langle P_V^+ \xi, P_V^+ \xi \rangle_{W^{1,2}} - \langle P_V^- \xi, P_V^- \xi \rangle_{W^{1,2}} + \tilde{\mathcal{L}}^{*\circ}(P_V^0 \xi) \quad \forall \xi \in \mathbf{B}_\eta(H_V).$$

Now Corollary A.2 gives rise to

$$C_q(\mathcal{L}^*, \gamma_0; \mathbb{K}) = C_q(\tilde{\mathcal{L}}^*, 0; \mathbb{K}) = C_{q-m^-(\gamma_0)}(\tilde{\mathcal{L}}^{*\circ}, 0; \mathbb{K}) \quad \forall q = 0, 1, \dots, \quad (4.23)$$

and $C_*(\tilde{\mathcal{L}}^{*\circ}, 0; \mathbb{K}) = C_*(\tilde{\mathcal{L}}^\circ, 0; \mathbb{K})$ by (4.22). From (4.23) and Theorem 3.8 we arrive at the following shifting theorem.

Theorem 4.4 $C_q(\mathcal{L}, \gamma_0; \mathbb{K}) = C_q(\tilde{\mathcal{L}}, 0; \mathbb{K}) = C_{q-m^-(\gamma_0)}(\tilde{\mathcal{L}}^\circ, 0; \mathbb{K}) \quad \forall q = 0, 1, \dots$

By Theorem A.5 there exists a ball $\mathbf{B}_\mu(X_V) \subset \mathbf{B}_\tau(X_V)$, an origin-preserving local homeomorphism $\tilde{\varphi}$ from $\mathbf{B}_\mu(X_V)$ to an open neighborhood of 0 in X_V with $\tilde{\varphi}(\mathbf{B}_\mu(X_V)) \subset \mathbf{B}_r(X_V)$ such that

$$\tilde{\mathcal{L}}^*|_{X_V} \circ \tilde{\varphi}(\xi) = \frac{1}{2} \langle \tilde{B}^*(0)\xi^\perp, \xi^\perp \rangle_{W^{1,2}} + \tilde{\mathcal{L}}^*(\tilde{h}(\xi^0) + \xi^0) \quad (4.24)$$

for any $\xi \in \mathbf{B}_\mu(X_V)$, where $\xi^0 = P^0(\xi)$ and $\xi^\perp = \xi - \xi^0$. Since $\mu \leq \tau$, from (4.7) and (4.18)-(4.19) we derive that (4.24) becomes:

$$\tilde{\mathcal{L}}|_{X_V} \circ \tilde{\varphi}(\xi) = \frac{1}{2} \langle \tilde{B}(0)\xi^\perp, \xi^\perp \rangle_{W^{1,2}} + \tilde{\mathcal{L}}^\circ(\xi^0) \quad \forall \xi \in \mathbf{B}_\mu(X_V). \quad (4.25)$$

Since $X_V^* = H_V^*$, $\star = 0, -$, as in the arguments below Theorem A.5 the following splitting theorem may be derived from (4.25) by changing $\mu > 0$ and $\tilde{\varphi}$ suitably.

Theorem 4.5 *Under the notations above, there exists a ball $\mathbf{B}_\mu(X_V) \subset \mathbf{B}_\tau(X_V)$, an origin-preserving local homeomorphism $\tilde{\varphi}$ from $\mathbf{B}_\mu(X_V)$ to an open neighborhood of 0 in X_V such that*

$$\tilde{\mathcal{L}}|_{X_V} \circ \tilde{\varphi}(\xi) = \frac{1}{2} \langle \tilde{B}(0)P^+\xi, P^+\xi \rangle_{W^{1,2}} - \|P^-\xi\|_{W^{1,2}}^2 + \tilde{\mathcal{L}}^\circ(\xi^0) \quad (4.26)$$

for any $\xi \in \mathbf{B}_\mu(X_V)$, where $\xi^\star = P^\star\xi$, $\star = -, 0, +$ and $\tilde{\mathcal{L}}^\circ$ is as in (4.21).

Remark 4.6 It is easily shown that $W_V^{1,2}(I, \mathbb{R}^n) = W_0^{1,2}(I, \mathbb{R}^n)$ and $C_V^1(I, \mathbb{R}^n) = C_0^1(I, \mathbb{R}^n)$ if M_0 and M_1 are two disjoint points. For the latter case along the proof lines of [16, Th.5.1] Caponio-Javaloyes-Masiello proved (4.26) in [11, Th.7] and hence the shifting theorem $C_*(\tilde{\mathcal{L}}|_{X_V}, 0; \mathbb{K}) = C_{*-m^-(\gamma_0)}(\tilde{\mathcal{L}}^\circ, 0; \mathbb{K})$. They also claimed that (29) of [11], or equivalently $C_*(\tilde{\mathcal{L}}, 0; \mathbb{K}) = C_*(\tilde{\mathcal{L}}|_{X_V}, 0; \mathbb{K})$, can be obtained with Palais' theorems 16 and 17 in [41] as in [17]. A detailed proof of such a claim is not trivial and was recently given in [12] with Chang's ideas of [14]. Different from [14] the gradient of $\tilde{\mathcal{L}}$ is not of the type identity plus a compact operator and thus the deformation retracts yielded by the flow of it cannot be assured to be also continuous in X_V . A new technique was introduced to overcome this difficulty.

Step 2. *Complete the proofs of Theorem 1.2, 1.3.*

Note that the differential at 0 of the chart Φ in (4.2),

$$d\Phi(0) : H_V \rightarrow T_{\gamma_0}\Lambda_N(M) = W^{1,2}(\gamma_0^*TM), \quad \xi \mapsto \sum_{i=1}^n \xi_i e_i$$

is a Hilbert space isomorphism and

$$\text{EXP}_{\gamma_0}(d\Phi(0)\xi)(t) = \exp_{\gamma_0(t)}((d\Phi(0)\xi)(t)) = \Phi(\xi)(t)$$

for $\xi \in \mathbf{B}_{2r}(H_V) := \{\xi \in H_V \mid \|\xi\|_1 < 2r\}$. That is, $\text{EXP}_{\gamma_0} \circ d\Phi(0) = \Phi$ on $\mathbf{B}_{2r}(H_V)$. Since $\tilde{\mathcal{L}} = \mathcal{L} \circ \Phi$ on $\mathbf{B}_{2r}(H_V)$ by (4.4), we get

$$\mathcal{L} \circ \text{EXP}_{\gamma_0} \circ d\Phi(0) = \mathcal{L} \circ \Phi = \tilde{\mathcal{L}} \quad \text{on } \mathbf{B}_{2r}(H_V) \quad (4.27)$$

and thus $\nabla(\mathcal{L} \circ \text{EXP}_{\gamma_0})(d\Phi(0)\xi) = d\Phi(0)\nabla\tilde{\mathcal{L}}(\xi)$ for all $\xi \in \mathbf{B}_{2r}(H_V)$. It follows that

$$\mathcal{A}(d\Phi(0)\xi) = d\Phi(0)\tilde{A}(\xi) \quad \forall \xi \in \mathbf{B}_{2r}(H_V) \cap X_V, \quad (4.28)$$

$$D\mathcal{A}(0) \circ d\Phi(0) = d\Phi(0) \circ D\tilde{A}(0) \quad (4.29)$$

because the Hilbert space isomorphism $d\Phi(0) : H_V \rightarrow T_{\gamma_0}\Lambda_N(M)$ induces a Banach space isomorphism from X_V to $T_{\gamma_0}\mathcal{X} = T_{\gamma_0}C_N^1(I, M)$. (4.29) implies that $d\Phi(0)(H_V^\star) = \mathbf{H}^\star(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0))$ for $\star = -, 0, +$, and so

$$d\Phi(0) \circ P_V^\star = P^\star \circ d\Phi(0), \quad \star = -, 0, +. \quad (4.30)$$

Shrinking $\delta > 0$ above (1.6) so that $\delta < \tau$, (4.30), (4.28) and (4.6) lead to

$$\begin{aligned} 0 &= d\Phi(0) \circ (I - P_V^0)\tilde{A}(d\Phi(0)^{-1}\zeta + \tilde{h}(d\Phi(0)^{-1}\zeta)) \\ &= (I - P^0) \circ d\Phi(0) \circ \tilde{A} \circ d\Phi(0)^{-1}(\zeta + d\Phi(0) \circ \tilde{h}(d\Phi(0)^{-1}\zeta)) \\ &= (I - P^0) \circ \mathcal{A}(\zeta + d\Phi(0) \circ \tilde{h}(d\Phi(0)^{-1}\zeta)) \end{aligned}$$

for $\zeta \in \mathbf{B}_\delta(\mathbf{H}^0(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)))$. Recall that we have obtained $(I - P^0)\mathcal{A}(\zeta + h(\zeta)) = 0$ for any $\zeta \in \mathbf{B}_\delta(\mathbf{H}^0(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)))$ above (1.6). By the uniqueness of h there, $h(\zeta) = d\Phi(0) \circ \tilde{h}(d\Phi(0)^{-1}\zeta)$ and hence

$$\begin{aligned} \mathcal{L}^\circ(d\Phi(0)\xi) &= \mathcal{L} \circ \text{EXP}_{\gamma_0}(d\Phi(0)\xi + h(d\Phi(0)\xi)) \\ &= \mathcal{L} \circ \text{EXP}_{\gamma_0} \circ d\Phi(0)(\xi + d\Phi(0)^{-1} \circ h(d\Phi(0)\xi)) \\ &= \mathcal{L} \circ \text{EXP}_{\gamma_0} \circ d\Phi(0)(\xi + \tilde{h}(\xi)) \\ &= \tilde{\mathcal{L}}(\xi + \tilde{h}(\xi)) = \tilde{\mathcal{L}}^\circ(\xi) \quad \forall \xi \in \mathbf{B}_\delta(H_V^0) \end{aligned} \quad (4.31)$$

by (1.6), (4.27) and the definition of $\tilde{\mathcal{L}}^\circ$ in (4.21). Hence $C_*(\mathcal{L}^\circ, 0; \mathbb{K}) = C_*(\tilde{\mathcal{L}}^\circ, 0; \mathbb{K})$. (That is, this and Theorem 4.4 give the first equality in Theorem 1.4 too.)

In order to prove Theorem 1.2, by Claims 3.5, 3.7 and Theorem 3.8 it suffices to prove (iii)-(iv). Note that $d\Phi(0)(\mathbf{B}_\delta(H_V)) = \mathbf{B}_\delta(T_{\gamma_0}\Lambda_N(M))$ and (4.29) implies that

$$\begin{aligned} \langle P_V^\star \xi, P_V^\star \xi \rangle_{W^{1,2}} &= \langle d\Phi(0) \circ P_V^\star \xi, d\Phi(0) \circ P_V^\star \xi \rangle_1 \\ &= \langle P^\star \circ d\Phi(0)\xi, P^\star \circ d\Phi(0)\xi \rangle_1 \end{aligned} \quad (4.32)$$

for $\xi \in \mathbf{B}_\delta(H_V) \subset \mathbf{B}_\tau(H_V)$ and $\star = +, -$. Moreover, as in (4.27) we have

$$\mathcal{L}^* \circ \text{EXP}_{\gamma_0} \circ d\Phi(0) = \mathcal{L}^* \circ \Phi = \tilde{\mathcal{L}}^* \quad \text{on } \mathbf{B}_{2r}(H_V).$$

Define $\psi : \mathbf{B}_\delta(T_{\gamma_0}\Lambda_N(M)) \rightarrow T_{\gamma_0}\Lambda_N(M)$ by $\psi = d\Phi(0) \circ \tilde{\psi} \circ [d\Phi(0)]^{-1}$. For $\zeta \in \mathbf{B}_\delta(T_{\gamma_0}\Lambda_N(M))$ and $\xi = [d\Phi(0)]^{-1}\zeta$ we get

$$\begin{aligned} \mathcal{L}^* \circ \text{EXP}_{\gamma_0} \circ \psi(\zeta) &= \mathcal{L}^* \circ \text{EXP}_{\gamma_0} \circ d\Phi(0) \circ \tilde{\psi} \circ [d\Phi(0)]^{-1}\zeta = \tilde{\mathcal{L}}^* \circ \tilde{\psi}(\xi), \\ \mathcal{L}^\circ(P^0\zeta) &= \mathcal{L}^\circ(P^0 \circ d\Phi(0)\xi) = \mathcal{L}^\circ(d\Phi(0) \circ P_V^0\xi) = \tilde{\mathcal{L}}^\circ(P_V^0\xi) \end{aligned}$$

by (4.31). These, (4.32) and Theorem 4.3 give Theorem 1.2(iii) after shrinking $\delta > 0$ such that $\delta < \eta$.

As to Theorem 1.2(iv), for any open neighborhood W of 0 in $U = \mathbf{B}_{2r}(H_V)$ and a field \mathbb{K} , writing $W_X = W \cap X_V$ as an open subset of X_V and using Theorem A.9 we deduce that the inclusion

$$\left(\tilde{\mathcal{L}}_c^* \cap W_X, \tilde{\mathcal{L}}_c^* \cap W_X \setminus \{0\}\right) \hookrightarrow \left(\tilde{\mathcal{L}}_c^* \cap W, \tilde{\mathcal{L}}_c^* \cap W \setminus \{0\}\right)$$

induces isomorphisms

$$H_*\left(\tilde{\mathcal{L}}_c^* \cap W_X, \tilde{\mathcal{L}}_c^* \cap W_X \setminus \{0\}; \mathbb{K}\right) \cong H_*\left(\tilde{\mathcal{L}}_c^* \cap W, \tilde{\mathcal{L}}_c^* \cap W \setminus \{0\}; \mathbb{K}\right).$$

The expected conclusion follows from this immediately.

Finally, let us prove Theorem 1.3. Since $d\Phi(0)$ is a Banach isomorphism from X_V to $C_{TN}^1(\gamma_0^*TM)$ we may choose $\epsilon > 0$ such that

$$[d\Phi(0)]^{-1}(\mathbf{B}_\epsilon(C_{TN}^1(\gamma_0^*TM))) \subset \mathbf{B}_\mu(X_V).$$

By (4.29), $D\mathcal{A}(0) \circ d\Phi(0) = d\Phi(0) \circ D\tilde{A}(0) = d\Phi(0) \circ \tilde{B}(0)$. So for $\zeta \in \mathbf{B}_\epsilon(C_{TN}^1(\gamma_0^*TM))$ and $\xi = [d\Phi(0)]^{-1}\zeta$ we have

$$\begin{aligned} d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)(\zeta^+, \zeta^+) &= d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0)(P^+\zeta, P^+\zeta) \\ &= \langle D\mathcal{A}(0)P^+\zeta, P^+\zeta \rangle_1 \\ &= \langle D\mathcal{A}(0) \circ d\Phi(0) \circ P_V^+\xi, d\Phi(0) \circ P_V^+\xi \rangle_1 \\ &= \langle \tilde{B}(0)P_V^+\xi, P_V^+\xi \rangle_{W^{1,2}} = \langle \tilde{B}(0)\xi^+, \xi^+ \rangle_{W^{1,2}}, \\ \|P^-\zeta\|_{W^{1,2}}^2 &= (P^-\zeta, P^-\zeta)_{W^{1,2}} = \langle P^- \circ d\Phi(0)\xi, P^- \circ d\Phi(0)\xi \rangle_{W^{1,2}} \\ &= \langle d\Phi(0) \circ P_V^-\xi, d\Phi(0) \circ P_V^-\xi \rangle_{W^{1,2}} \\ &= \langle P_V^-\xi, P_V^-\xi \rangle_1 = \|P_V^-\xi\|_1^2 \end{aligned}$$

by (4.30). Define $\varphi : \mathbf{B}_\epsilon(C_{TN}^1(\gamma_0^*TM)) \rightarrow C_{TN}^1(\gamma_0^*TM)$ by $\varphi = d\Phi(0) \circ \tilde{\varphi} \circ [d\Phi(0)]^{-1}$. As above these and Theorem 4.5 yield Theorem 1.3. \square

5 Proof of Theorems 1.5, 1.6, 1.7, 1.8

5.1 Stronger versions of Theorems 1.5, 1.6, 1.7

Write $\mathcal{H} = \Lambda M$ for conveniences. Let $\gamma_0 \in \mathcal{H}$ be a nonconstant critical point of \mathcal{L} such that $S^1 \cdot \gamma_0^m$ is an isolated critical orbits of \mathcal{L} on \mathcal{H} for some integer $m \in \mathbb{N}$. Write $\mathcal{O} = S^1 \cdot \gamma_0$ as before. The m -th iteration map φ_m defined in (1.21) is an embedding and it also satisfies $\varphi_m(\mathcal{O}) = S^1 \cdot \gamma_0^m$ and

$$\left. \begin{aligned} \mathcal{L} \circ \varphi_m &= m^2 \mathcal{L} \quad \text{and} \\ \varphi_m([ms] \cdot \alpha) &= [s] \cdot \varphi_m(\alpha) \quad \forall \alpha \in \mathcal{H}, [s] \in S^1. \end{aligned} \right\} \quad (5.1)$$

Clearly, φ_m induces a bundle embedding $\tilde{\varphi}_m : T\mathcal{H} \rightarrow T\mathcal{H}$ given by

$$(x, v) \mapsto (x^m, v^m), \quad (5.2)$$

where $v^m(t) = v(mt)$ for $t \in S^1$. Consider the equivalent Riemannian-Hilbert structure on $T\mathcal{H}$:

$$\langle \xi, \eta \rangle_m = m^2 \int_0^1 \langle \xi(t), \eta(t) \rangle dt + \int_0^1 \langle \nabla_\gamma^h \xi(t), \nabla_\gamma^h \eta(t) \rangle dt. \quad (5.3)$$

Then $\varphi_m : (\mathcal{H}, \langle \cdot, \cdot \rangle_1) \rightarrow (\mathcal{H}, \langle \cdot, \cdot \rangle_m)$ is an isometry up to the factor m^2 , that is,

$$\langle \tilde{\varphi}_m(\xi), \tilde{\varphi}_m(\eta) \rangle_m = m^2 \langle \xi, \eta \rangle_1 \quad \forall \xi, \eta \in T\mathcal{H}.$$

Denote by $\hat{N}\varphi_m(\mathcal{O})$ the normal bundle of $\varphi_m(\mathcal{O})$ with respect to this metric. Its fibre $\hat{N}\varphi_m(\mathcal{O})_{\gamma_0^m}$ at γ_0^m is the orthogonal complementary of $\dot{\gamma}_0^m \mathbb{R}$ in $(T_{\gamma_0^m} \mathcal{H}, \langle \cdot, \cdot \rangle_m)$. Note that $\hat{N}\varphi_m(\mathcal{O}) = N\mathcal{O}$ if $m = 1$. As in (3.18) we define

$$\hat{\mathcal{F}}^\tau : \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\delta) \rightarrow \mathbb{R}, (y, v) \mapsto \mathcal{L}^\tau \circ \text{EXP}(y, v) \quad (5.4)$$

by shrinking $\delta > 0$ if necessary. It is C^{2-0} , and satisfies the (PS) condition and

$$\hat{\mathcal{F}}^\tau([s] \cdot y, [s] \cdot v) = \hat{\mathcal{F}}^\tau(y, v) \quad \forall (y, v) \in \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\delta), \forall [s] \in S^1.$$

We also write $\hat{\mathcal{F}} := \hat{\mathcal{F}}^0$ and $\hat{\mathcal{F}}^* := \hat{\mathcal{F}}^1$. Let

$$\begin{aligned} X\hat{N}\varphi_m(\mathcal{O}) &= T_{\varphi_m(\mathcal{O})}\mathcal{X} \cap \hat{N}\varphi_m(\mathcal{O}), \\ \hat{N}\varphi_m(\mathcal{O})(\varepsilon) &= \{(x, v) \in \hat{N}\varphi_m(\mathcal{O}) \mid \|v\|_{W^{1,2}} < \varepsilon\}, \\ X\hat{N}\varphi_m(\mathcal{O})(\varepsilon) &= \{(x, v) \in X\hat{N}\varphi_m(\mathcal{O}) \mid \|v\|_{C^1} < \varepsilon\} \end{aligned}$$

for $\varepsilon > 0$. They become the sets in (1.11) for $m = 1$. If $\varepsilon > 0$ is sufficiently small, the map EXP in (1.12) restricts to a diffeomorphism from the normal disk bundle $\hat{N}\varphi_m(\mathcal{O})(\varepsilon)$ onto an open neighborhood $\hat{\mathcal{N}}(\varphi_m(\mathcal{O}), \varepsilon)$ of $\varphi_m(\mathcal{O})$ in \mathcal{H} . Denote by $\hat{N}\varphi_m(\mathcal{O})(\varepsilon)_x$ and $X\hat{N}\varphi_m(\mathcal{O})(\varepsilon)_x$ the fibers of $\hat{N}\varphi_m(\mathcal{O})(\varepsilon)$ and $X\hat{N}\varphi_m(\mathcal{O})(\varepsilon)$ at $x \in \varphi_m(\mathcal{O})$, respectively. Let $\hat{\nabla}$ denote the gradient with respect to the metric in (5.3), and let

$$\hat{A}_x \text{ be the restriction of the gradient } \hat{\nabla}\hat{\mathcal{F}} \text{ to } \hat{N}\varphi_m(\mathcal{O})(\varepsilon)_x \cap X\hat{N}\varphi_m(\mathcal{O})_x.$$

It is clear that

$$\hat{A}_{s \cdot x}(s \cdot v) = s \cdot \hat{A}_x(v) \quad \forall s \in S^1, v \in \hat{N}\varphi_m(\mathcal{O})(\varepsilon)_x \cap X\hat{N}\varphi_m(\mathcal{O})_x. \quad (5.5)$$

If $\delta > 0$ is small enough \hat{A}_x is C^1 in $X\hat{N}\varphi_m(\mathcal{O})(\delta)_x$ (and so $\mathcal{L} \circ \text{EXP}|_{X\hat{N}\varphi_m(\mathcal{O})(\delta)_x}$ is C^2) by (5.12), (5.44), (5.42), (5.39) and (B.14). Denote by \hat{B}_x the symmetric bilinear form $d^2(\mathcal{L} \circ \text{EXP}|_{X\hat{N}\varphi_m(\mathcal{O})(\varepsilon)_x})(0_x)$. By (i) above Claim 5.9 we see that it can be extended into such a form on $N\varphi_m(\mathcal{O})_x$, also denoted by \hat{B}_x , whose associated self-adjoint operator is Fredholm, and has finite dimensional negative definite and null spaces $\mathbf{H}^-(\hat{B}_x)$ and $\mathbf{H}^0(\hat{B}_x)$ such that $\mathbf{H}^-(\hat{B}_x) + \mathbf{H}^0(\hat{B}_x)$ is contained in $X\hat{N}\varphi_m(\mathcal{O})_x$. With respect to the metric in (5.3) we have an orthogonal decomposition

$$\hat{N}\varphi_m(\mathcal{O})_x = \mathbf{H}^-(\hat{B}_x) \hat{\oplus} \mathbf{H}^0(\hat{B}_x) \hat{\oplus} \mathbf{H}^+(\hat{B}_x)$$

(according to the negative definite, null and positive of \hat{B}_x), and hence a natural C^2 Hilbert vector bundle orthogonal decomposition

$$\hat{N}\varphi_m(\mathcal{O}) = \mathbf{H}^-(\hat{B}) \hat{\oplus} \mathbf{H}^0(\hat{B}) \hat{\oplus} \mathbf{H}^+(\hat{B}) \quad (5.6)$$

with $\mathbf{H}^\star(\hat{B})_x = \mathbf{H}^\star(\hat{B}_x)$ for $x \in \varphi_m(\mathcal{O})$ and $\star = +, 0, -$ because $\hat{B}_{s \cdot x}(s \cdot \xi, s \cdot \eta) = \hat{B}_x(\xi, \eta)$ for any $s \in S^1$ and $x \in \varphi_m(\mathcal{O})$. Clearly, (5.6) induces a C^2 Banach vector bundle (topological) direct sum decomposition

$$X\hat{N}\varphi_m(\mathcal{O}) = \mathbf{H}^-(\hat{B}) \hat{+} \mathbf{H}^0(\hat{B}) \hat{+} (\mathbf{H}^+(\hat{B}) \cap X\hat{N}\varphi_m(\mathcal{O})). \quad (5.7)$$

Note: If the symbol $\hat{\cdot}$ in all notations since (5.3) are moved out we understand the corresponding results to be with respect to the metric in (1.1).

Since both $\dim \mathbf{H}^-(B_x)$ and $\dim \mathbf{H}^-(\hat{B}_x)$ (resp. $\dim \mathbf{H}^0(B_x)+1$ and $\dim \mathbf{H}^0(\hat{B}_x)+1$) are equal to the Morse index (resp. nullity) of the symmetric bilinear form $d^2(\mathcal{L} \circ \text{EXP}|_{T_x\mathcal{X}})(0_x)$, we obtain

$$\text{rank} \mathbf{H}^-(B) = \text{rank} \mathbf{H}^-(\hat{B}) \quad \text{and} \quad \text{rank} \mathbf{H}^0(B) = \text{rank} \mathbf{H}^0(\hat{B}). \quad (5.8)$$

That is, they are equal to Morse index $m^-(\varphi_m(\mathcal{O}))$ and nullity $m^0(\varphi_m(\mathcal{O}))$ of $\varphi_m(\mathcal{O})$, respectively. Let $\hat{\mathbf{P}}^\star$ be the orthogonal bundle projections from $\hat{N}\varphi_m(\mathcal{O})$ onto $\mathbf{H}^\star(\hat{B})$, $\star = +, 0, -$, and let $\mathbf{H}^0(\hat{B})(\epsilon) = \mathbf{H}^0(\hat{B}) \cap \hat{N}\varphi_m(\mathcal{O})(\epsilon)$ for $\epsilon > 0$. Note that $\mathbf{H}^0(\hat{B})(\epsilon) \subset X\hat{N}\varphi_m(\mathcal{O})$ and that for any $\delta > 0$ we may choose $\epsilon > 0$ so small that $\mathbf{H}^0(\hat{B})(\sqrt{m}\epsilon) \subset X\hat{N}\varphi_m(\mathcal{O})(\delta)$ since $\mathbf{H}^0(\hat{B})$ has finite rank. By the implicit function theorem, if $\epsilon > 0$ is sufficiently small for each $x \in \varphi_m(\mathcal{O})$ there exists a unique C^1 map

$$\hat{\mathbf{h}}_x : \mathbf{H}^0(\hat{B})(\sqrt{m}\epsilon)_x \rightarrow \mathbf{H}^-(\hat{B})_x \hat{+} (\mathbf{H}^+(\hat{B})_x \cap X\hat{N}\varphi_m(\mathcal{O})_x) \quad (5.9)$$

satisfying $\hat{\mathbf{h}}_x(0_x) = 0_x$ and

$$(\hat{\mathbf{P}}_x^+ + \hat{\mathbf{P}}_x^-) \circ \hat{A}_x(v + \hat{\mathbf{h}}_x(v)) = 0 \quad \forall v \in \mathbf{H}^0(\hat{B})(\sqrt{m}\epsilon)_x. \quad (5.10)$$

By (5.5) the map $\hat{\mathbf{h}}_x$ is also S_x^1 -equivariant. Define the functional

$$\hat{\mathcal{L}}_\Delta^\circ : \mathbf{H}^0(\hat{B})(\sqrt{m}\epsilon) \ni (x, v) \rightarrow \mathcal{L} \circ \text{EXP}_x(v + \hat{\mathbf{h}}_x(v)) \in \mathbb{R}. \quad (5.11)$$

It is C^1 and has the isolated critical orbit $\varphi_m(\mathcal{O})$. By (5.12), (5.44), (5.42), (5.39) and (B.14) we may see that the restriction of $\hat{\mathcal{L}}_\Delta^\circ$ to each fiber $\mathbf{H}^0(B)(\sqrt{m}\epsilon)_x$, denoted by $\hat{\mathcal{L}}_{\Delta x}^\circ$, is C^2 . The following theorem includes Theorem 1.5 as a special case (by taking $m = 1$).

Theorem 5.1 *Under the above notations, there exists a C^k convex quadratic growth Lagrangian $L^* : TM \rightarrow \mathbb{R}$ such that (i) in Theorem 1.5 and the following hold:*

- (ii) *The corresponding functional \mathcal{L}^* in (1.24) is C^{2-0} in \mathcal{H} . All functional $\mathcal{L}^\tau = (1-\tau)\mathcal{L} + \tau\mathcal{L}^*$, $\tau \in [0, 1]$, have only a critical orbit $\varphi_m(\mathcal{O})$ in some neighborhood of $\varphi_m(\mathcal{O}) \subset \mathcal{H}$, and each \mathcal{L}^τ satisfies the Palais-Smale condition. Moreover, $C_*(\mathcal{L}^*, \varphi_m(\mathcal{O}); \mathbb{K}) = C_*(\mathcal{L}, \varphi_m(\mathcal{O}); \mathbb{K})$.*

- (iii) By shrinking the above $\epsilon > 0$ (if necessary) there exist a S^1 -invariant open neighborhood \hat{U} of the zero section of $\hat{N}\varphi_m(\mathcal{O})$, a S^1 -equivariant fiber-preserving, continuous and fibrewise C^1 map \hat{h} given by (5.9) and (5.10), and a S^1 -equivariant fiber-preserving homeomorphism $\hat{\Upsilon} : \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon) \rightarrow \hat{U}$ such that

$$\hat{\mathcal{F}}^* \circ \hat{\Upsilon}(u) = \|\hat{\mathbf{P}}^+ u\|_m^2 - \|\hat{\mathbf{P}}^- u\|_m^2 + \hat{\mathcal{L}}_\Delta^\circ(\hat{\mathbf{P}}^0 u)$$

for all $u \in \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon)$, where $\|\xi\|_m^2 = \langle \xi, \xi \rangle_m$. Moreover,

$$\hat{\Upsilon}((\mathbf{P}^- + \mathbf{P}^0)\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon)) \subset X\hat{N}\varphi_m(\mathcal{O}),$$

and $\hat{\Upsilon} : (\mathbf{P}^- + \mathbf{P}^0)\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon) \rightarrow \hat{\Upsilon}((\mathbf{P}^- + \mathbf{P}^0)\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon))$ is also a homeomorphism even if the topology on the latter is taken as the induced one by $X\hat{N}\varphi_m(\mathcal{O})$. (This implies that $\hat{N}\varphi_m(\mathcal{O})$ and $X\hat{N}\varphi_m(\mathcal{O})$ induce the same topology in $\hat{\Upsilon}((\mathbf{P}^- + \mathbf{P}^0)\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon))$.)

- (iv) Let $\hat{\mathcal{F}}^{*X}$ be the restriction of $\hat{\mathcal{F}}^*$ to $\hat{N}(\tilde{\varphi}_m(\mathcal{O}))(\sqrt{m}\delta) \cap T_{\varphi_m(\mathcal{O})}\mathcal{X}$. For any open neighborhood \hat{W} of $\varphi_m(\mathcal{O})$ in $\hat{N}(\tilde{\varphi}_m(\mathcal{O}))(\sqrt{m}\delta)$, write $\hat{W}_X := \hat{W} \cap T_{\varphi_m(\mathcal{O})}\mathcal{X}$ as an open subset of $T_{\varphi_m(\mathcal{O})}\mathcal{X}$, then the inclusion

$$\left((\hat{\mathcal{F}}^{*X})_e \cap \hat{W}_X, (\hat{\mathcal{F}}^{*X})_e \cap \hat{W}_X \setminus \varphi_m(\mathcal{O}) \right) \hookrightarrow \left((\hat{\mathcal{F}}^*)_e \cap \hat{W}, (\hat{\mathcal{F}}^*)_e \cap \hat{W} \setminus \varphi_m(\mathcal{O}) \right),$$

where $e = m^2c = m^2\mathcal{L}(\gamma_0)$, induces isomorphisms from

$$H_* \left((\hat{\mathcal{F}}^{*X})_e \cap \hat{W}_X, (\hat{\mathcal{F}}^{*X})_e \cap \hat{W}_X \setminus \varphi_m(\mathcal{O}); \mathbb{K} \right)$$

to $H_* \left((\hat{\mathcal{F}}^*)_e \cap \hat{W}, (\hat{\mathcal{F}}^*)_e \cap \hat{W} \setminus \varphi_m(\mathcal{O}); \mathbb{K} \right)$, where \mathbb{K} is a field. Moreover, the corresponding conclusion can be obtained if

$$\left((\hat{\mathcal{F}}^{*X})_e \cap \hat{W}_X, (\hat{\mathcal{F}}^{*X})_e \cap \hat{W}_X \setminus \varphi_m(\mathcal{O}) \right) \text{ and } \left((\hat{\mathcal{F}}^*)_e \cap \hat{W}, (\hat{\mathcal{F}}^*)_e \cap \hat{W} \setminus \varphi_m(\mathcal{O}) \right),$$

are replaced by $((\hat{\mathcal{F}}^{*X})_e^\circ \cap \hat{W}_X \cup \varphi_m(\mathcal{O}), (\hat{\mathcal{F}}^{*X})_e^\circ \cap \hat{W}_X)$ and $((\hat{\mathcal{F}}^*)_e^\circ \cap \hat{W} \cup \varphi_m(\mathcal{O}), (\hat{\mathcal{F}}^*)_e^\circ \cap \hat{W})$, respectively, where $(\hat{\mathcal{F}}^{*X})_e^\circ = \{\hat{\mathcal{F}}^{*X} < e\}$ and $(\hat{\mathcal{F}}^*)_e^\circ = \{\hat{\mathcal{F}}^* < e\}$.

- (v) If $\hat{\mathcal{F}}^{*X}$ and $\hat{\mathcal{F}}^*$ in (iv) are replaced by $\hat{\mathcal{F}}^X$ and $\hat{\mathcal{F}}$, respectively, then the corresponding conclusion also holds true, where $\hat{\mathcal{F}}^X$ is the restriction of $\hat{\mathcal{F}}$ to $\hat{N}(\tilde{\varphi}_m(\mathcal{O}))(\sqrt{m}\delta) \cap T_{\varphi_m(\mathcal{O})}\mathcal{X}$.

By Section 3, for example, Claim 3.9 and Theorem 3.10, Theorem 5.1(i)-(ii) and hence Theorem 1.5(i)-(ii) are clear. It is not hard to see that (iv) and (v) imply Theorem 1.5(iv) and (v), respectively.

The following splitting lemma includes Theorem 1.6.

Theorem 5.2 *Under the above notations, by shrinking the above $\epsilon > 0$ (if necessary) there exist a S^1 -invariant open neighborhood \hat{V} of the zero section of $X\hat{N}\varphi_m(\mathcal{O})$, a S^1 -equivariant fiber-preserving, continuous and fibrewise C^1 map \hat{h} given by (5.9) and*

(5.10), and a S^1 -equivariant fiber-preserving homeomorphism $\widehat{\Psi} : X\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon) \rightarrow \widehat{V}$ such that

$$\hat{\mathcal{F}}^X \circ \widehat{\Psi}(x, v) = \frac{1}{2}d^2\mathcal{L}|_{\mathcal{X}}(x)(\hat{\mathbf{P}}_x^+v, \hat{\mathbf{P}}_x^+v) - \|\hat{\mathbf{P}}_x^-v\|_m^2 + \hat{\mathcal{L}}_{\Delta x}^\circ(\hat{\mathbf{P}}_x^0v)$$

for all $(x, v) \in X\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon)$.

Clearly, $\hat{\mathcal{L}}_{\Delta x}^\circ$ is S_x^1 -invariant. Let $C_*(\hat{\mathcal{L}}_{\Delta x}^\circ, 0; \mathbb{K})^{S_x^1}$ denote the subgroup of all elements in $C_*(\hat{\mathcal{L}}_{\Delta x}^\circ, 0; \mathbb{K})$, which are fixed by the induced action of S_x^1 on the homology. The following is a generalization of Theorem 1.7.

Theorem 5.3 *Let \mathbb{K} be a field of characteristic 0 or prime to order $|S_{\gamma_0^m}^1|$ of $S_{\gamma_0^m}^1$. Then for any $x \in \varphi_m(\mathcal{O})$ and $q \in \mathbb{N} \cup \{0\}$,*

$$\begin{aligned} & C_q(\mathcal{L}, S^1 \cdot \gamma_0^m; \mathbb{K}) \\ &= \left(H_{m^-(S^1 \cdot \gamma_0^m)}(\mathbf{H}^-(\hat{B})_x, \mathbf{H}^-(\hat{B})_x \setminus \{0_x\}; \mathbb{K}) \otimes C_{q-m^-(S^1 \cdot \gamma_0^m)-1}(\hat{\mathcal{L}}_{\Delta x}^\circ, 0; \mathbb{K}) \right)^{S_x^1} \\ &\oplus \left(H_{m^-(S^1 \cdot \gamma_0^m)}(\mathbf{H}^-(\hat{B})_x, \mathbf{H}^-(\hat{B})_x \setminus \{0_x\}; \mathbb{K}) \otimes C_{q-m^-(S^1 \cdot \gamma_0^m)}(\hat{\mathcal{L}}_{\Delta x}^\circ, 0; \mathbb{K}) \right)^{S_x^1} \end{aligned}$$

provided $m^-(S^1 \cdot \gamma_0^m)m^0(S^1 \cdot \gamma_0^m) > 0$. Moreover,

$$C_q(\hat{\mathcal{L}}, S^1 \cdot \gamma_0^m; \mathbb{K}) = (C_{q-1}(\hat{\mathcal{L}}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1} \oplus (C_q(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1}$$

if $m^-(S^1 \cdot \gamma_0^m) = 0$ and $m^0(S^1 \cdot \gamma_0^m) > 0$, and

$$\begin{aligned} C_q(\hat{\mathcal{L}}, S^1 \cdot \gamma_0^m; \mathbb{K}) &= H_q(\mathbf{H}^-(\hat{B}), \mathbf{H}^-(\hat{B}) \setminus \varphi_m(\mathcal{O}); \mathbb{K}) \\ &= \left(H_{q-1}(\mathbf{H}^-(\hat{B})_x, \mathbf{H}^-(\hat{B})_x \setminus \{0_x\}; \mathbb{K}) \right)^{S_x^1} \\ &\oplus \left(H_q(\mathbf{H}^-(\hat{B})_x, \mathbf{H}^-(\hat{B})_x \setminus \{0_x\}; \mathbb{K}) \right)^{S_x^1} \end{aligned}$$

if $m^-(S^1 \cdot \gamma_0^m) > 0$ and $m^0(S^1 \cdot \gamma_0^m) = 0$. Finally, $C_q(\hat{\mathcal{L}}, S^1 \cdot \gamma_0^m; \mathbb{K}) = H_q(S^1; \mathbb{K})$ for any Abelian group \mathbb{K} if $m^-(S^1 \cdot \gamma_0^m) = m^0(S^1 \cdot \gamma_0^m) = 0$.

The proof of this theorem does not need Theorem 5.2.

When \mathcal{L} is replaced by \mathcal{L}^* , we have the corresponding \hat{A}_x^* and \hat{B}_x^* for $x \in \varphi_m(\mathcal{O})$, and hence the C^2 Hilbert vector bundle orthogonal decomposition and C^1 Banach vector bundle (topological) direct sum decomposition as in (5.6) and (5.7),

$$\begin{aligned} \hat{N}\varphi_m(\mathcal{O}) &= \mathbf{H}^-(\hat{B}^*) \hat{\oplus} \mathbf{H}^0(\hat{B}^*) \hat{\oplus} \mathbf{H}^+(\hat{B}^*), \\ X\hat{N}\varphi_m(\mathcal{O}) &= \mathbf{H}^-(\hat{B}^*) \hat{\oplus} \mathbf{H}^0(\hat{B}^*) \hat{\oplus} (\mathbf{H}^+(\hat{B}^*) \cap X\hat{N}\varphi_m(\mathcal{O})). \end{aligned}$$

(Since $\mathbf{H}^-(\hat{B}^*) \hat{\oplus} \mathbf{H}^0(\hat{B}^*) \subset X\hat{N}\varphi_m(\mathcal{O})$ has finite rank the bundles $\hat{N}\varphi_m(\mathcal{O})$ and $X\hat{N}\varphi_m(\mathcal{O})$ induce an equivalent topology on it. See [33, Lemma 3.15] for a proof.) Let $\hat{\mathbb{P}}^\star$ be the corresponding bundle projections from $\hat{N}\varphi_m(\mathcal{O})$ onto $\mathbf{H}^\star(B^*)$, $\star = +, 0, -$. By Corollary 2.3 we have

$$\hat{A}_x^*(v) = \hat{A}_x(v) \quad \forall (x, v) \in XN\mathcal{O}(\varepsilon)_x \quad (5.12)$$

(shrinking $\varepsilon > 0$ if necessary), and hence $\hat{B}_x^* = \hat{B}_x$ for any $x \in \varphi_m(\mathcal{O})$. The latter implies that

$$\mathbf{H}^*(\hat{B}^*) = \mathbf{H}^*(\hat{B}) \quad \text{and} \quad \hat{\mathbf{P}}^* = \hat{\mathbb{P}}^*, \quad \star = +, 0, -. \quad (5.13)$$

It follows from this, (5.12) and (5.10) that the C^1 map $\hat{\mathbf{h}}_x$ in (5.9) satisfies

$$(\hat{\mathbb{P}}_x^+ + \hat{\mathbb{P}}_x^-) \circ \hat{A}_x^*(v + \hat{\mathbf{h}}_x(v)) = 0 \quad \forall v \in \mathbf{H}^0(B)(\sqrt{m}\epsilon)_x.$$

Define C^1 the functional

$$\hat{\mathcal{L}}_{\Delta}^{*\circ} : \mathbf{H}^0(\hat{B}^*)(\sqrt{m}\epsilon) \ni (x, v) \rightarrow \mathcal{L}^* \circ \text{EXP}_x(v + \hat{\mathbf{h}}_x(v)) \in \mathbb{R}, \quad (5.14)$$

which has the isolated critical orbit $\varphi_m(\mathcal{O})$. Later we shall prove that the restriction of $\hat{\mathcal{L}}_{\Delta}^{*\circ}$ to each fiber $\mathbf{H}^0(\hat{B})(\sqrt{m}\epsilon)_x$, denoted by $\hat{\mathcal{L}}_{\Delta x}^{*\circ}$, is C^2 . Clearly, $\hat{\mathcal{L}}_{\Delta}^{*\circ}$ is S_x^1 -invariant and has an isolated critical point 0_x . By Corollary 2.3 and (5.11) we have

$$\begin{aligned} \hat{\mathcal{L}}_{\Delta x}^{*\circ}(v) &= \mathcal{L}^* \circ \text{EXP}_x(v + \hat{\mathbf{h}}_x(v)) \\ &= \mathcal{L} \circ \text{EXP}_x(v + \hat{\mathbf{h}}_x(v)) = \hat{\mathcal{L}}_{\Delta x}^{\circ}(v) \end{aligned} \quad (5.15)$$

for any $v \in \mathbf{H}^0(\hat{B})(\sqrt{m}\epsilon)_x = \mathbf{H}^0(\hat{B}^*)(\sqrt{m}\epsilon)_x$ (shrinking $\epsilon > 0$ if necessary). Hence

$$C_q(\hat{\mathcal{L}}_{\Delta x}^{\circ}, 0; \mathbb{K}) = C_q(\hat{\mathcal{L}}_{\Delta x}^{*\circ}, 0; \mathbb{K}) \quad (5.16)$$

for any $x \in \varphi_m(\mathcal{O})$ and $q = 0, 1, \dots$.

As below Lemma 4.2, for every $x \in \varphi_m(\mathcal{O})$ there exists a continuous map

$$\hat{\mathbf{B}}_x : X\hat{N}\varphi_m(\mathcal{O})(\delta)_x \rightarrow L_s(\hat{N}\varphi_m(\mathcal{O})_x)$$

such that for any $\zeta \in X\hat{N}\varphi_m(\mathcal{O})(\delta)_x$ and $\xi, \eta \in \hat{N}\varphi_m(\mathcal{O})_x$,

$$(D\hat{A}_{\gamma_0^m}^*(\zeta)\xi, \eta)_{W^{1,2}} = d^2\hat{\mathcal{F}}_{\gamma_0^m}^{*X}(\zeta)(\xi, \eta) = (\hat{\mathbf{B}}_x^*(\zeta)\xi, \eta)_{W^{1,2}}.$$

Clearly, $\hat{\mathbf{B}}_x^*(0) = \hat{B}_x^*$. The proof of the following proposition will be postponed to Section 5.3.

Proposition 5.4 *($\hat{N}\varphi_m(\mathcal{O})_{\gamma_0^m}, X\hat{N}\varphi_m(\mathcal{O})_{\gamma_0^m}, \hat{\mathcal{F}}_{\gamma_0^m}^*, \hat{A}_{\gamma_0^m}^*, \hat{\mathbf{B}}_{\gamma_0^m}^*$) satisfies the conditions of Theorem A.1 (and hence Theorem A.5) around the critical point $\gamma_0^m \equiv 0_{\gamma_0^m}$.*

The notations in this subsection and following subsection seem to be complex a bit, but they shall be convenient for Section 6.

5.2 Proposition 5.4 leading to Theorems 5.1, 5.2, 5.3

Clearly, Proposition 5.4 implies that $(\hat{N}\varphi_m(\mathcal{O})_x, X\hat{N}\varphi_m(\mathcal{O})_x, \hat{\mathcal{F}}_x^*, \hat{A}_x^*, \hat{\mathbf{B}}_x^*)$ satisfies the conditions of Theorems A.1, A.5 around 0_x for each $x \in \varphi_m(\mathcal{O})$. (Indeed, let $[s] \in S^1$ such that $x = [s] \cdot \gamma_0^m$. Using the Hilbert isomorphism $\hat{N}\varphi_m(\mathcal{O})_{\gamma_0^m} \rightarrow \hat{N}\varphi_m(\mathcal{O})_x$ and Theorems A.4, A.8 ones easily prove them.) By Theorem A.1 and

(5.12) we obtain a S_x^1 -invariant open neighborhood \hat{U}_x of 0_x in $\hat{N}\varphi_m(\mathcal{O})(\delta)_x$ and a S_x^1 -equivariant origin-preserving homeomorphism

$$\hat{\Upsilon}_x : \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon)_x \rightarrow \hat{U}_x \quad (5.17)$$

(shrinking $\epsilon \in (0, 1)$ if necessary), such that

$$\hat{\mathcal{F}}_x^* \circ \hat{\Upsilon}_x(u) = \|\hat{\mathbf{P}}_x^+ u\|_m^2 - \|\hat{\mathbf{P}}_x^- u\|_m^2 + \hat{\mathcal{F}}_x^*(\hat{\mathbf{P}}_x^0 u + \hat{\mathbf{h}}_x(\hat{\mathbf{P}}_x^0 u)) \quad (5.18)$$

for all $u \in \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon)_x$. $\hat{\Upsilon}_x$ also maps $(\hat{\mathbf{P}}_x^- + \hat{\mathbf{P}}_x^0)\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon)_x$ into $X\hat{N}\varphi_m(\mathcal{O})_x$ and

$$\hat{\Upsilon}_x : (\hat{\mathbf{P}}_x^- + \hat{\mathbf{P}}_x^0)\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon)_x \rightarrow \hat{\Upsilon}_x \left((\hat{\mathbf{P}}_x^- + \hat{\mathbf{P}}_x^0)\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon)_x \right)$$

is a homeomorphism even if the topology on the latter is taken as the induced one by $X\hat{N}\varphi_m(\mathcal{O})_x$. Moreover, the S_x^1 -invariant functional

$$\mathbf{H}^0(\hat{B}_x^*)(\sqrt{m}\epsilon) \ni z \mapsto \hat{\mathcal{F}}_x^{*\circ}(z) := \hat{\mathcal{F}}_x^*(z + \hat{\mathbf{h}}_x(z)) \quad (5.19)$$

is C^2 , has the isolated critical point $0_x \in \mathbf{H}^0(\hat{B}_x^*)$ and $d^2 \hat{\mathcal{F}}_x^{*\circ}(0_x) = 0$. Obverse that

$$\begin{aligned} \hat{\mathcal{L}}_{\Delta x}^{*\circ}(v) &= \hat{\mathcal{L}}^* \circ \text{EXP}_x(v + \hat{\mathbf{h}}_x(v)) \\ &= \hat{\mathcal{F}}^*|_{\hat{N}\varphi_m(\mathcal{O})(\delta)_x}(v + \hat{\mathbf{h}}_x(v)) \\ &= \hat{\mathcal{F}}_x^{*\circ}(v) \quad \forall v \in \mathbf{H}^0(\hat{B}_x^*)(\sqrt{m}\epsilon) \end{aligned} \quad (5.20)$$

because of (5.14) and the definition of $\hat{\mathcal{F}}^*$ in (5.4).

Let \hat{U} be a S^1 -invariant tubular open neighborhood of the zero section of $\hat{N}\varphi_m(\mathcal{O})$. It is a fibre bundle over $\varphi_m(\mathcal{O})$. By shrinking \hat{U} we may require that the fiber of \hat{U} at x is given by \hat{U}_x in (5.17). Define maps $\hat{\Upsilon} : \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon) \rightarrow \hat{U}$ and

$$\hat{\mathbf{h}} : \mathbf{H}^0(\hat{B}^*)(\sqrt{m}\epsilon) \rightarrow \mathbf{H}^-(\hat{B}^*) \hat{\oplus} (\mathbf{H}^+(\hat{B}^*) \cap X\hat{N}\varphi_m(\mathcal{O}))$$

by $\hat{\Upsilon}|_{\hat{N}\varphi_m(\mathcal{O})_x(\sqrt{m}\epsilon)} = \hat{\Upsilon}_x$ and $\hat{\mathbf{h}}|_{\mathbf{H}^0(\hat{B}^*)_x(\sqrt{m}\epsilon)} = \hat{\mathbf{h}}_x$ for any $x \in \varphi_m(\mathcal{O})$, respectively. Here both $\hat{\mathbf{h}}_x$ and $\hat{\Upsilon}_x$ are given by (5.9) and (5.17), respectively. Since

$$\begin{aligned} \hat{A}_{s \cdot x}^*(s \cdot v) &= s \cdot \hat{A}_x^*(v) \quad \forall s \in S^1, v \in XN\varphi_m(\mathcal{O})(\epsilon)_x, \\ \hat{B}_{s \cdot x}^*(s \cdot \xi, s \cdot \eta) &= \hat{B}_x^*(\xi, \eta) \quad \forall s \in S^1, x \in \varphi_m(\mathcal{O}), \xi, \eta \in N\varphi_m(\mathcal{O})_x, \end{aligned}$$

as in the proofs of Theorem 7.3 and Corollary 7.1 in [16, page 72] it follows from these, (5.9)-(5.10) and (5.15) that $\hat{\mathbf{h}}$ is a S^1 -equivariant fiber-preserving, continuous and fibrewise C^1 map and that $\hat{\Upsilon}$ is a S^1 -equivariant fiber-preserving homeomorphism from $\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon)$ onto \hat{U} . So we obtain the first claim in Theorem 5.1(iii). The second may be derived from either the above arguments or the corresponding conclusion in Theorem A.2 of [33, 34].

By completely similar arguments we may use Theorem A.5 to derive Theorem 5.2. Now we prove Theorem 5.1(iv)-(v) and Theorem 5.3.

Firstly, we prove the latter. If $m^0(S^1 \cdot \gamma_0^m) = m^-(S^1 \cdot \gamma_0^m) = 0$, by Theorem 5.1(iii) we have $\hat{\mathcal{F}}^* \circ \hat{\Upsilon}(u) = m^2c + \|u\|_m^2$ for all $u \in \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon)$, and hence

$$\begin{aligned} & C_*(\hat{\mathcal{L}}, S^1 \cdot \gamma_0^m; \mathbb{K}) = C_*(\hat{\mathcal{F}}^*, S^1 \cdot \gamma_0^m; \mathbb{K}) = C_*(\hat{\mathcal{F}}^* \circ \hat{\Upsilon}, S^1 \cdot \gamma_0^m; \mathbb{K}) \\ &= H_*(\{\hat{\mathcal{F}}^* \circ \hat{\Upsilon} < m^2c\} \cup S^1 \cdot \gamma_0^m, \{\hat{\mathcal{F}}^* \circ \hat{\Upsilon} < m^2c\}; \mathbb{K}) = H_*(S^1 \cdot \gamma_0^m; \mathbb{K}) \\ &= H_*(S^1; \mathbb{K}) \quad \text{because we have assume } \gamma_0 \text{ to be nonconstant.} \end{aligned}$$

If either $m^0(S^1 \cdot \gamma_0^m) = 0$ and $m^-(S^1 \cdot \gamma_0^m) > 0$ or $m^0(S^1 \cdot \gamma_0^m) > 0$ and $m^-(S^1 \cdot \gamma_0^m) = 0$ ones easily see the desired conclusions from the following proof in the case $m^0(S^1 \cdot \gamma_0^m) > 0$ and $m^-(S^1 \cdot \gamma_0^m) > 0$. For the proof of the final case it suffices to prove:

Claim 5.5 *Let \mathbb{K} be a field of characteristic 0 or prime to order $|S_{\gamma_0^m}^1|$ of $S_{\gamma_0^m}^1$. For any $x \in \varphi_m(\mathcal{O})$ and $q = 0, 1, \dots$, it holds that*

$$\begin{aligned} & C_q(\hat{\mathcal{F}}^*, \varphi_m(\mathcal{O}); \mathbb{K}) \\ &= \left(H_{m^-(S^1 \cdot \gamma_0^m)}(\mathbf{H}^-(\hat{B})_x, \mathbf{H}^-(\hat{B})_x \setminus \{0_x\}; \mathbb{K}) \otimes C_{q-m^-(S^1 \cdot \gamma_0^m)}(\hat{\mathcal{F}}_x^{*\circ}, 0; \mathbb{K}) \right)^{S_x^1} \\ &\oplus \left(H_{m^-(S^1 \cdot \gamma_0^m)}(\mathbf{H}^-(\hat{B})_x, \mathbf{H}^-(\hat{B})_x \setminus \{0_x\}; \mathbb{K}) \otimes C_{q-m^-(S^1 \cdot \gamma_0^m)-1}(\hat{\mathcal{F}}_x^{*\circ}, 0; \mathbb{K}) \right)^{S_x^1}. \end{aligned}$$

Proof. Since the map $S^1 \times \hat{N}\varphi_m(\mathcal{O})_x \rightarrow \hat{N}\varphi_m(\mathcal{O})$ with $(s, v) \mapsto s \cdot v$ is a normal covering with group of covering transformations S_x^1 ([25, page 500]), so is the map $S^1 \times \hat{W}_x \rightarrow \hat{W}$ with $(s, v) \mapsto s \cdot v$ for any subset $\hat{W} \subset \hat{N}\varphi_m(\mathcal{O})$ with properties

$$\hat{W}_{s \cdot x} = s \cdot \hat{W}_x \quad \forall x \in \varphi_m(\mathcal{O}), s \in S^1, \quad (5.21)$$

where $\hat{W}_x = \hat{W} \cap \hat{N}\varphi_m(\mathcal{O})_x$, Then $\hat{W} = (S^1 \times \hat{W}_x)/S_x^1$, where S_x^1 acts on $S^1 \times \hat{W}_x$ by covering transformations as described above.

Recalling $\mathcal{L}|_{\mathcal{O}} = c$ let $e = m^2c = \hat{\mathcal{F}}^*|_{\varphi_m(\mathcal{O})}$. Since $\varphi_m(\mathcal{O})$ is compact and $\epsilon \in (0, 1)$, there exists an integer $l > 1$ such that $\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon^l)_x \subset \hat{U}_x$ and hence

$$\hat{\Upsilon}_x^{-1} \left(\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon^l)_x \right) \subset \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon)_x.$$

Then it is easily seen that

$$\hat{W} := \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon^l), \quad \hat{W} \cap \{\hat{\mathcal{F}}^* \leq e\} \quad \text{and} \quad (\hat{W} \setminus \varphi_m(\mathcal{O})) \cap \{\hat{\mathcal{F}}^* \leq e\}$$

satisfy (5.21). It follows that

$$\begin{aligned} & C_q(\hat{\mathcal{F}}^*, \varphi_m(\mathcal{O}); \mathbb{K}) = H_q \left(\hat{W} \cap \{\hat{\mathcal{F}}^* \leq e\}, (\hat{W} \setminus \varphi_m(\mathcal{O})) \cap \{\hat{\mathcal{F}}^* \leq e\}; \mathbb{K} \right) \\ &= H_q \left((S^1 \times (\hat{W}_x \cap \{\hat{\mathcal{F}}_x^* \leq e\}))/S_x^1, (S^1 \times ((\hat{W}_x \setminus \{0_x\}) \cap \{\hat{\mathcal{F}}_x^* \leq e\}))/S_x^1; \mathbb{K} \right) \\ &= H_q((S^1 \times \Delta_x)/S_x^1, (S^1 \times \Delta'_x)/S_x^1; \mathbb{K}), \end{aligned} \quad (5.22)$$

where

$$\begin{aligned} \Delta_x &= \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon^l)_x \cap \{\hat{\mathcal{F}}_x^* \circ \hat{\Upsilon}_x \leq e\}, \\ \Delta'_x &= (\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon^l)_x \setminus \{0_x\}) \cap \{\hat{\mathcal{F}}_x^* \circ \hat{\Upsilon}_x \leq e\}. \end{aligned}$$

The final equality in (5.22) comes from the fact that $\hat{\Upsilon}_x^{-1}$ is a S_x^1 -equivariant homeomorphism from

$$(S^1 \times (\hat{W}_x \cap \{\hat{\mathcal{F}}_x^* \leq e\}), S^1 \times ((\hat{W}_x \setminus \{0_x\}) \cap \{\hat{\mathcal{F}}_x^* \leq e\}))$$

to $(S^1 \times \Delta_x, S^1 \times \Delta'_x)$ by (5.18). Let

$$\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon^l)_x^{-0} := \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon^l)_x \cap (\mathbf{H}^-(\hat{B})_x^- \hat{\oplus} \mathbf{H}^0(\hat{B})_x),$$

which is a finite dimensional C^3 -smooth manifold contained in $X\hat{N}\varphi_m(\mathcal{O})$. Define

$$(\hat{\mathcal{F}}_x^* \circ \hat{\Upsilon}_x)^{-0} : \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon^l)_x^{-0} \rightarrow \mathbb{R} \quad (5.23)$$

by $(\hat{\mathcal{F}}_x^* \circ \hat{\Upsilon}_x)^{-0}(v^- + v^0) = -\|v^-\|_m^2 + \hat{\mathcal{F}}_x^*(v^0 + \hat{\mathfrak{h}}_x(v^0)) = -\|v^-\|_m^2 + \hat{\mathcal{F}}_x^{*\circ}(v^0)$. It is C^2 because of (5.19). Observe that $(S^1 \times \Delta_x, S^1 \times \Delta'_x)$ can be retracted S_x^1 -equivariantly into $(S^1 \times \Delta_x^{-0}, S^1 \times \Delta'_x{}^{-0})$, where

$$\begin{aligned} \Delta_x^{-0} &= \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon^l)_x^{-0} \cap \{(\hat{\mathcal{F}}_x^* \circ \hat{\Upsilon}_x)^{-0} \leq e\}, \\ \Delta'_x{}^{-0} &= (\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon^l)_x^{-0} \setminus \{0_x\}) \cap \{(\hat{\mathcal{F}}_x^* \circ \hat{\Upsilon}_x)^{-0} \leq e\}. \end{aligned}$$

From this and (5.22) we derive that

$$C_q(\hat{\mathcal{F}}^*, \varphi_m(\mathcal{O}); \mathbb{K}) = H_q((S^1 \times \Delta_x^{-0})/S_x^1, (S^1 \times \Delta'_x{}^{-0})/S_x^1; \mathbb{K}). \quad (5.24)$$

Using either Satz 6.6 on the page 57 of [42] or the proof of Lemma 3.6 in [2] (with Theorem 7.2 on the page 142 of [7]) we know that the transfer is an isomorphism

$$H_*((S^1 \times \Delta_x^{-0})/S_x^1, (S^1 \times \Delta'_x{}^{-0})/S_x^1; \mathbb{K}) = H_*((S^1 \times \Delta_x^{-0}, S^1 \times \Delta'_x{}^{-0}); \mathbb{K})^{S_x^1}.$$

This, (5.24) and the Künneth formula lead to

$$C_q(\hat{\mathcal{F}}^*, \varphi_m(\mathcal{O}); \mathbb{K}) = H_{q-1}(\Delta_x^{-0}, \Delta'_x{}^{-0}; \mathbb{K})^{S_x^1} \oplus H_q(\Delta_x^{-0}, \Delta'_x{}^{-0}; \mathbb{K})^{S_x^1}. \quad (5.25)$$

As in the proof of Theorem 5.5 on page 51 of [16] we may obtain

$$\begin{aligned} &H_q(\Delta_x^{-0}, \Delta'_x{}^{-0}; \mathbb{K}) \\ &= H_{m^-(\varphi_m(\mathcal{O}))}(\mathbf{H}^-(\hat{B})(\sqrt{m}\epsilon)_x, \partial\mathbf{H}^-(\hat{B})(\sqrt{m}\epsilon)_x; \mathbb{K}) \otimes C_{q-m^-(\varphi_m(\mathcal{O}))}(\hat{\mathcal{F}}_x^{*\circ}, 0_x; \mathbb{K}) \\ &= H_{m^-(\varphi_m(\mathcal{O}))}(\mathbf{H}^-(\hat{B})_x, \mathbf{H}^-(\hat{B})_x \setminus \{0_x\}; \mathbb{K}) \otimes C_{q-m^-(\varphi_m(\mathcal{O}))}(\hat{\mathcal{F}}_x^{*\circ}, 0_x; \mathbb{K}) \end{aligned}$$

for all $q = 0, 1, \dots$. (See the proof of Lemma 4.6 in [31]). This and (5.25) give Claim 5.5 immediately. \square

Now by Theorem 5.1(ii) for $q \in \mathbb{N} \cup \{0\}$ we have

$$C_q(\mathcal{L}, \varphi_m(\mathcal{O}); \mathbb{K}) = C_q(\mathcal{L}^*, \varphi_m(\mathcal{O}); \mathbb{K}) = C_q(\hat{\mathcal{F}}^*, \varphi_m(\mathcal{O}); \mathbb{K}).$$

Moreover, (5.15) and (5.20) imply $\hat{\mathcal{L}}_{\Delta_x}^\circ(v) = \hat{\mathcal{L}}_{\Delta_x}^{*\circ}(v) = \hat{\mathcal{F}}_x^{*\circ}(v)$ for any $x \in \mathcal{O}$ and $v \in \mathbf{H}^0(\hat{B})(\sqrt{m}\epsilon)_x = \mathbf{H}^0(\hat{B}^*)(\sqrt{m}\epsilon)_x$. Theorem 5.3 follows from these and Claim 5.5 immediately.

Another proof of Claim 5.5. It follows Theorem 5.2 that for any $q \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} C_q(\hat{\mathcal{F}}^X, \varphi_m(\mathcal{O}); \mathbb{K}) &= C_q(\hat{\mathcal{F}}^X \circ \hat{\Psi}, \varphi_m(\mathcal{O}); \mathbb{K}) \\ &= C_q((\hat{\mathcal{F}}^X \circ \hat{\Psi})^{-0}, \varphi_m(\mathcal{O}); \mathbb{K}), \end{aligned} \quad (5.26)$$

where $(\hat{\mathcal{F}}^X \circ \hat{\Psi})^{-0} : \hat{N}\varphi_m(\mathcal{O})^{-0}(\rho) = \hat{N}\varphi_m(\mathcal{O})(\rho) \cap (\mathbf{H}^-(\hat{B}) \hat{\oplus} \mathbf{H}^0(\hat{B})) \rightarrow \mathbb{R}$ is defined by

$$(\hat{\mathcal{F}}^X \circ \hat{\Psi})^{-0}(x, v) = -\|\hat{\mathbf{P}}_x^- v\|_m^2 + \hat{\mathcal{L}}_{\Delta x}^\circ(\hat{\mathbf{P}}_x^0 v) \quad (5.27)$$

for very small $\rho > 0$, and the second equality in (5.26) is obtained by the standard deformation method as done above (3.6) of [33]. Taking \hat{W} to be

$$\{(\hat{\mathcal{F}}^X \circ \hat{\Psi})^{-0} \leq m^2 c\} \quad \text{or} \quad \hat{W} = \{(\hat{\mathcal{F}}^X \circ \hat{\Psi})^{-0} \leq m^2 c\} \setminus \{\varphi_m(\mathcal{O})\}$$

we have $\hat{W} = (S^1 \times \hat{W}_x)/S_x^1$ and hence

$$\begin{aligned} &C_q((\hat{\mathcal{F}}^X \circ \hat{\Psi})^{-0}, \varphi_m(\mathcal{O}); \mathbb{K}) \\ &= H_q\left((S^1 \times \{(\hat{\mathcal{F}}^X \circ \hat{\Psi})_x^{-0} \leq m^2 c\})/S_x^1, (S^1 \times (\{(\hat{\mathcal{F}}^X \circ \hat{\Psi})_x^{-0} \leq m^2 c\} \setminus \{0_x\}))/S_x^1; \mathbb{K}\right). \end{aligned}$$

Almost repeating the arguments below (5.24) in the proof of Claim 5.5 we arrive at

Claim 5.6 *Let \mathbb{K} be a field of characteristic 0 or prime to order $|S_{\gamma_0^m}^1|$ of $S_{\gamma_0^m}^1$. For any $x \in \varphi_m(\mathcal{O})$ and $q = 0, 1, \dots$, it holds that*

$$\begin{aligned} &C_q(\hat{\mathcal{F}}^X, \varphi_m(\mathcal{O}); \mathbb{K}) \\ &= \left(H_{m-(S^1 \cdot \gamma_0^m)}(\mathbf{H}^-(\hat{B})_x, \mathbf{H}^-(\hat{B})_x \setminus \{0_x\}; \mathbb{K}) \otimes C_{q-m-(S^1 \cdot \gamma_0^m)}(\hat{\mathcal{L}}_{\Delta x}^\circ, 0; \mathbb{K})\right)^{S_x^1} \\ &\oplus \left(H_{m-(S^1 \cdot \gamma_0^m)}(\mathbf{H}^-(\hat{B})_x, \mathbf{H}^-(\hat{B})_x \setminus \{0_x\}; \mathbb{K}) \otimes C_{q-m-(S^1 \cdot \gamma_0^m)-1}(\hat{\mathcal{L}}_{\Delta x}^\circ, 0; \mathbb{K})\right)^{S_x^1}. \end{aligned}$$

As the reasoning of (5.26), using Theorem 5.1(iii) we derive

$$C_q(\hat{\mathcal{F}}^*, \varphi_m(\mathcal{O}); \mathbb{K}) = C_q(\hat{\mathcal{F}}^* \circ \hat{\Upsilon}, \varphi_m(\mathcal{O}); \mathbb{K}) = C_q((\hat{\mathcal{F}}^* \circ \hat{\Upsilon})^{-0}, \varphi_m(\mathcal{O}); \mathbb{K})$$

for each $q \in \mathbb{N} \cup \{0\}$, where $(\hat{\mathcal{F}}^* \circ \hat{\Upsilon})^{-0}$ is defined by (5.23). By (5.20) and (5.15), $\hat{\mathcal{F}}^{*\circ}(v) = \hat{\mathcal{L}}_{\Delta}^{\circ}(v) = \hat{\mathcal{L}}_{\Delta}^\circ(v)$ for $v \in \hat{N}\varphi_m(\mathcal{O})^{-0}(\rho)$ if $\rho > 0$ is small enough. So

$$(\hat{\mathcal{F}}^* \circ \hat{\Upsilon})^{-0}(x, v^- + v^0) = -\|v^-\|_m^2 + \hat{\mathcal{L}}_{\Delta x}^\circ(v^0) = (\hat{\mathcal{F}}^X \circ \hat{\Psi})^{-0}(x, v^- + v^0)$$

for $(x, v^- + v^0) \in \hat{N}\varphi_m(\mathcal{O})^{-0}(\rho)$ by (5.27). It follows that

$$C_q((\hat{\mathcal{F}}^* \circ \hat{\Upsilon})^{-0}, \varphi_m(\mathcal{O}); \mathbb{K}) = C_q((\hat{\mathcal{F}}^X \circ \hat{\Psi})^{-0}, \varphi_m(\mathcal{O}); \mathbb{K})$$

and thus

$$C_q(\hat{\mathcal{F}}^*, \varphi_m(\mathcal{O}); \mathbb{K}) = C_q(\hat{\mathcal{F}}^X, \varphi_m(\mathcal{O}); \mathbb{K}) \quad \forall q \in \mathbb{N} \cup \{0\}$$

by (5.26). This and Claim 5.6 lead to Claim 5.5. \square

Theorem 1.5(i) and the excision property of relative homology groups imply $C_q(\hat{\mathcal{F}}^{*X}, \varphi_m(\mathcal{O}); \mathbb{K}) = C_q(\hat{\mathcal{F}}^X, \varphi_m(\mathcal{O}); \mathbb{K}) \quad \forall q \in \mathbb{N} \cup \{0\}$.

Claim 5.7 *If \mathbb{K} is a field, for each $q \in \mathbb{N} \cup \{0\}$ the group*

$$C_q(\hat{\mathcal{F}}^{*X}, \varphi_m(\mathcal{O}); \mathbb{K}) = C_q(\hat{\mathcal{F}}^*, \varphi_m(\mathcal{O}); \mathbb{K})$$

is a finite dimensional vector space over \mathbb{K} .

Proof. As in the proof of [8, (7.15)] we use Proposition A.1 in [8] to derive that

$$\begin{aligned} & \dim C_q(\hat{\mathcal{F}}^*, \varphi_m(\mathcal{O}); \mathbb{K}) \\ & \leq 2 \left(\dim C_{q-m-}(S^1, \gamma_0^m)(\hat{\mathcal{F}}_x^{*\circ}, 0; \mathbb{K}) + \dim C_{q-m-}(S^1, \gamma_0^m)_{-1}(\hat{\mathcal{F}}_x^{*\circ}, 0; \mathbb{K}) \right) \end{aligned}$$

for any $x \in \varphi_m(\mathcal{O})$ and $q \in \mathbb{N} \cup \{0\}$. By [34, Remark 4.6] or [33, Remark 2.24] we know that $C_q(\hat{\mathcal{F}}_x^{*\circ}, 0; \mathbb{K})$ is finite dimensional and $\dim C_q(\hat{\mathcal{F}}_x^{*\circ}, 0; \mathbb{K}) = 0$ for almost all q . Of course, this claim can also be proved with the method therein. \square

Claim 5.8 *For any open neighborhood \hat{W} of $\varphi_m(\mathcal{O})$ in $\hat{N}(\tilde{\varphi}_m(\mathcal{O}))(\sqrt{m}\delta)$, write $\hat{W}_X = \hat{W} \cap T_{\varphi_m(\mathcal{O})}\mathcal{X}$ as an open subset of $T_{\varphi_m(\mathcal{O})}\mathcal{X}$, then the inclusion*

$$\left((\hat{\mathcal{F}}^{*X})_e \cap \hat{W}_X, (\hat{\mathcal{F}}^{*X})_e \cap \hat{W}_X \setminus \varphi_m(\mathcal{O}) \right) \hookrightarrow \left((\hat{\mathcal{F}}^*)_e \cap \hat{W}, (\hat{\mathcal{F}}^*)_e \cap \hat{W} \setminus \varphi_m(\mathcal{O}) \right),$$

where $e = m^2c = m^2\mathcal{L}(\gamma_0)$, induces surjective homomorphisms from

$$H_* \left((\hat{\mathcal{F}}^{*X})_e \cap \hat{W}_X, (\hat{\mathcal{F}}^{*X})_e \cap \hat{W}_X \setminus \varphi_m(\mathcal{O}); \mathbb{K} \right)$$

to $H_ \left((\hat{\mathcal{F}}^*)_e \cap \hat{W}, (\hat{\mathcal{F}}^*)_e \cap \hat{W} \setminus \varphi_m(\mathcal{O}); \mathbb{K} \right)$ for any Abelian group \mathbb{K} .*

This may be obtained by Corollary 3.3 of [33]. For clearness we present its proof with the proof method of [34, Cor.2.5] since it shows that we need not to assume the normal bundle of \mathcal{O} to be trivial in Theorem 3.17 of [33].

Proof of Claim 5.8. If the closure of \hat{W} in $\hat{N}(\tilde{\varphi}_m(\mathcal{O}))$ is contained in the interior of $\hat{N}(\tilde{\varphi}_m(\mathcal{O}))(\sqrt{m}\delta)$, it is easy to show that the closure of \hat{W}_X in $T_{\varphi_m(\mathcal{O})}\mathcal{X}$ is also contained in the interior of $\hat{N}(\tilde{\varphi}_m(\mathcal{O}))(\sqrt{m}\delta) \cap T_{\varphi_m(\mathcal{O})}\mathcal{X}$ when the latter is equipped with topology of $T_{\varphi_m(\mathcal{O})}\mathcal{X}$. By the excision of relative homology groups we only need to prove Claim 5.8 for some open neighborhood \hat{W} of $\varphi_m(\mathcal{O})$ in $\hat{N}(\tilde{\varphi}_m(\mathcal{O}))(\sqrt{m}\delta)$.

Since $\mathbf{H}^-(\hat{B})(\nu) \hat{\oplus} \mathbf{H}^0(\hat{B})(\nu) \hat{\oplus} \mathbf{H}^+(\hat{B})(\nu) \subset \hat{N}_{\varphi_m(\mathcal{O})}(\sqrt{m}\epsilon)$ for $\nu \in (0, \sqrt{m}\epsilon/3)$, we may take

$$\hat{W} = \hat{\Upsilon}(\mathbf{H}^-(\hat{B})(\nu) \hat{\oplus} \mathbf{H}^0(\hat{B})(\nu) \hat{\oplus} \mathbf{H}^+(\hat{B})(\nu)), \quad \hat{V} = \hat{\Upsilon}(\mathbf{H}^-(\hat{B})(\nu) \hat{\oplus} \mathbf{H}^0(\hat{B})(\nu)).$$

Consider the deformation $\eta : \hat{W} \times [0, 1] \rightarrow \hat{W}$ given by

$$\eta(\hat{\Upsilon}(x, u^- + u^0 + u^+), t) = \hat{\Upsilon}(x, u^- + u^0 + tu^+).$$

It gives a deformation retract from $(\hat{\mathcal{F}}_e^* \cap \hat{W}, \hat{\mathcal{F}}_e^* \cap \hat{W} \setminus \varphi_m(\mathcal{O}))$ onto $(\hat{\mathcal{F}}_e^* \cap \hat{V}, \hat{\mathcal{F}}_e^* \cap \hat{V} \setminus \varphi_m(\mathcal{O}))$. Hence the inclusion

$$I : (\hat{\mathcal{F}}_e^* \cap \hat{V}, \hat{\mathcal{F}}_e^* \cap \hat{V} \setminus \varphi_m(\mathcal{O})) \hookrightarrow (\hat{\mathcal{F}}_e^* \cap \hat{W}, \hat{\mathcal{F}}_e^* \cap \hat{W} \setminus \varphi_m(\mathcal{O}))$$

induces isomorphisms between their relative singular groups. This means that for each nontrivial $\alpha \in H_q(\hat{\mathcal{F}}_e^* \cap \hat{W}, \hat{\mathcal{F}}_e^* \cap \hat{W} \setminus \varphi_m(\mathcal{O}); \mathbb{K})$ we can choose a relative singular cycle representative of it, $c = \sum_j g_j \sigma_j$, such that

$$|c| := \cup_j \sigma_j(\Delta^q) \subset \hat{\mathcal{F}}_e^* \cap \hat{V} \quad \text{and} \quad |\partial c| \subset \hat{\mathcal{F}}_e^* \cap \hat{V} \setminus \varphi_m(\mathcal{O}).$$

By the second conclusion of Theorem 5.1(iii) we deduce that c is also a relative singular cycle in $(\hat{\mathcal{F}}_e^{*X} \cap \hat{V}, \hat{\mathcal{F}}_e^{*X} \cap \hat{V} \setminus \varphi_m(\mathcal{O}))$, denoted by c_X for clearness. Precisely, if ι^V denote the identity map from $(\hat{\mathcal{F}}_e^{*X} \cap \hat{V}, \hat{\mathcal{F}}_e^{*X} \cap \hat{V} \setminus \varphi_m(\mathcal{O}))$ to $(\hat{\mathcal{F}}_e^* \cap \hat{V}, \hat{\mathcal{F}}_e^* \cap \hat{V} \setminus \varphi_m(\mathcal{O}))$ then $\iota^V(c_X) = c$. Let j^W denote the inclusion map from $(\hat{\mathcal{F}}_e^{*X} \cap \hat{W}, \hat{\mathcal{F}}_e^{*X} \cap \hat{W} \setminus \varphi_m(\mathcal{O}))$ to $(\hat{\mathcal{F}}_e^* \cap \hat{W}, \hat{\mathcal{F}}_e^* \cap \hat{W} \setminus \varphi_m(\mathcal{O}))$. Then I, ι^V, j^W and the inclusion

$$I^X : (\hat{\mathcal{F}}_e^{*X} \cap \hat{V}, \hat{\mathcal{F}}_e^{*X} \cap \hat{V} \setminus \varphi_m(\mathcal{O})) \hookrightarrow (\hat{\mathcal{F}}_e^{*X} \cap \hat{W}, \hat{\mathcal{F}}_e^{*X} \cap \hat{W} \setminus \varphi_m(\mathcal{O}))$$

satisfy $I \circ \iota^V = j^W \circ I^X$. Since $I_*([c]) = \alpha$ we obtain

$$\alpha = I_* \circ (\iota^V)_*[c_X] = (j^W)_* \circ (I^X)_*[c_X] = (j^W)_*((I^X)_*[c_X]).$$

Claim 5.8 is proved. \square

Now for a field \mathbb{K} those surjective homomorphisms in Claim 5.8 are isomorphisms because two sides are vector spaces of same finite dimension by Claim 5.7. This completes the proof of the first claim of Theorem 5.1(iv). Similarly, the second claim of Theorem 5.1(iv) can be proved as that of Claim 5.8. \square

Proof of Theorem 5.1(v). Take a small open neighborhood \mathcal{V} of $\varphi_m(\mathcal{O})$ in $T_{\varphi_m(\mathcal{O})}\mathcal{X}$ such that its closure is contained in the interior of \hat{W}_X and that $\hat{\mathcal{F}}^{*X}$ and $\hat{\mathcal{F}}^X$ are same in \mathcal{V} . Then

$$((\hat{\mathcal{F}}^{*X})_e \cap \mathcal{V}, (\hat{\mathcal{F}}^{*X})_e \cap \mathcal{V} \setminus \varphi_m(\mathcal{O})) = ((\hat{\mathcal{F}}^X)_e \cap \mathcal{V}, (\hat{\mathcal{F}}^X)_e \cap \mathcal{V} \setminus \varphi_m(\mathcal{O})). \quad (5.28)$$

By the excision of relative homology groups we deduce that the inclusion

$$((\hat{\mathcal{F}}^{*X})_e \cap \mathcal{V}, (\hat{\mathcal{F}}^{*X})_e \cap \mathcal{V} \setminus \varphi_m(\mathcal{O})) \hookrightarrow ((\hat{\mathcal{F}}^{*X})_e \cap \hat{W}_X, (\hat{\mathcal{F}}^{*X})_e \cap \hat{W}_X \setminus \varphi_m(\mathcal{O}))$$

induces isomorphisms between their homology groups. This and Theorem 5.1(iv) imply that the same claim holds true for the inclusion

$$I^{vw} : ((\hat{\mathcal{F}}^{*X})_e \cap \mathcal{V}, (\hat{\mathcal{F}}^{*X})_e \cap \mathcal{V} \setminus \varphi_m(\mathcal{O})) \hookrightarrow ((\hat{\mathcal{F}}^*)_e \cap \hat{W}, (\hat{\mathcal{F}}^*)_e \cap \hat{W} \setminus \varphi_m(\mathcal{O})). \quad (5.29)$$

By Corollary 2.3, $L^* \leq L$ and hence $\mathcal{L}^* \leq \mathcal{L}$ and $\hat{\mathcal{F}}^* \leq \hat{\mathcal{F}}$. The latter implies

$$((\hat{\mathcal{F}})_e \cap \hat{W}, (\hat{\mathcal{F}})_e \cap \hat{W} \setminus \varphi_m(\mathcal{O})) \subset ((\hat{\mathcal{F}}^*)_e \cap \hat{W}, (\hat{\mathcal{F}}^*)_e \cap \hat{W} \setminus \varphi_m(\mathcal{O})). \quad (5.30)$$

Moreover, by (5.28) we have also the inclusion

$$J^{vw} : ((\hat{\mathcal{F}}^X)_e \cap \mathcal{V}, (\hat{\mathcal{F}}^X)_e \cap \mathcal{V} \setminus \varphi_m(\mathcal{O})) \hookrightarrow ((\hat{\mathcal{F}})_e \cap \hat{W}, (\hat{\mathcal{F}})_e \cap \hat{W} \setminus \varphi_m(\mathcal{O})). \quad (5.31)$$

Hence $I^{vw} = h \circ J^{vw}$ and thus $I_*^{vw} = h_* \circ J_*^{vw}$, where h is the inclusion in (5.30). Since the homology groups of pairs in (5.29)-(5.31) are isomorphic vector spaces of

finite dimensions by Claim 5.7, I_*^{vw} are isomorphisms by Theorem 5.1(iv), both h_* and J_*^{vw} must be isomorphisms as well.

As above the excision leads to that the inclusion

$$((\hat{\mathcal{F}}^X)_e \cap \mathcal{V}, (\hat{\mathcal{F}}^X)_e \cap \mathcal{V} \setminus \varphi_m(\mathcal{O})) \hookrightarrow ((\hat{\mathcal{F}}^X)_e \cap \hat{W}_X, (\hat{\mathcal{F}}^X)_e \cap \hat{W}_X \setminus \varphi_m(\mathcal{O}))$$

induces isomorphisms between their homology groups. Composing the inclusion with

$$((\hat{\mathcal{F}}^X)_e \cap \hat{W}_X, (\hat{\mathcal{F}}^X)_e \cap \hat{W}_X \setminus \varphi_m(\mathcal{O})) \hookrightarrow ((\hat{\mathcal{F}})_e \cap \hat{W}, (\hat{\mathcal{F}})_e \cap \hat{W} \setminus \varphi_m(\mathcal{O})) \quad (5.32)$$

we get J^{vw} . It follows that the inclusion in (5.32) induces isomorphisms between their homology groups. Theorem 5.1(v) is proved. \square

5.3 Proof of Proposition 5.4

Since M is not necessarily orientable the bundle $\gamma_0^* TM \rightarrow S^1$ might be nontrivial. Define $E_\sigma := \text{diag}(\sigma, 1, \dots, 1) \in \mathbb{R}^{n \times n}$, where

$$\sigma = \sigma(\gamma_0) := \begin{cases} +1 & \text{if } \gamma_0^* TM \rightarrow S^1 \text{ is trivial,} \\ -1 & \text{else.} \end{cases}$$

Fix a Riemannian metric h on M and write $I = [0, 1]$. Viewing γ_0 as a 1-periodic C^k -map from \mathbb{R} to M , by parallel transport with respect to the Levi-Civita connection of h we can choose a unit orthonormal parallel frame field

$$\mathcal{T} = \mathcal{T}_\sigma : [0, 1] \rightarrow (\gamma_0|_I)^* TM, \quad t \mapsto (e_1(t), \dots, e_n(t))$$

such that $(e_1(1), \dots, e_n(1)) = (e_1(0), \dots, e_n(0))E_\sigma$. Clearly, such a field extends uniquely to E_σ -**1-invariant** unit orthonormal parallel frame field of along γ_0 ,

$$\tilde{\mathcal{T}}_\sigma : \mathbb{R} \ni t \mapsto \gamma_0^* TM, \quad t \mapsto (e_1(t), \dots, e_n(t)),$$

that is, it satisfies $(e_1(t+1), \dots, e_n(t+1)) = (e_1(t), \dots, e_n(t))E_\sigma \forall t \in \mathbb{R}$. All vectors in \mathbb{R}^n will be understood as row vectors without special statements.

From e_1, \dots, e_n , we may obtain a E_{σ^m} -1-invariant field along γ_0^m ,

$$(\tilde{e}_1(t), \dots, \tilde{e}_n(t)) := (e_1(mt), \dots, e_n(mt)),$$

that is, it satisfies $(\tilde{e}_1(t+1), \dots, \tilde{e}_n(t+1)) = (\tilde{e}_1(t), \dots, \tilde{e}_n(t))E_{\sigma^m} \forall t \in \mathbb{R}$.

A curve $\xi = (\xi_1, \dots, \xi_n) : \mathbb{R} \rightarrow \mathbb{R}^n$ is called E_{σ^m} -**1-invariant** if

$$\xi(t+1)^T = E_{\sigma^m} \xi(t)^T \quad \forall t \in \mathbb{R}.$$

Here $\xi(t)^T$ denotes the transpose of the matrix $\xi(t)$ as usual. Let $X_{\sigma^m} = C_{\sigma^m}^1(I, \mathbb{R}^n)$ be the set of all E_{σ^m} -1-invariant C^1 curves from \mathbb{R} to \mathbb{R}^n . It is a Banach space according to the usual C^1 -norm. Denote by

$$H_{\sigma^m} \quad \text{and} \quad \hat{H}_{\sigma^m}$$

the completions of X_{σ^m} with respect to inner products

$$\langle \xi, \eta \rangle_{1, \sigma^m} = \int_I [(\xi(t), \eta(t))_{\mathbb{R}^n} + (\dot{\xi}(t), \dot{\eta}(t))_{\mathbb{R}^n}] dt$$

and

$$\langle \xi, \eta \rangle_{\sigma^m} = \int_I [m^2(\xi(t), \eta(t))_{\mathbb{R}^n} + (\dot{\xi}(t), \dot{\eta}(t))_{\mathbb{R}^n}] dt \quad (5.33)$$

respectively. As sets both H_{σ^m} and \hat{H}_{σ^m} are same subsets of $W_{\sigma^m}^{1,2}(I, \mathbb{R}^n)$, i.e., E_{σ^m} -1-invariant $W_{\text{loc}}^{1,2}$ curves from \mathbb{R} to \mathbb{R}^n . Write

$$\|\xi\|_{1, \sigma^m} = \sqrt{\langle \xi, \xi \rangle_{1, \sigma^m}} \quad \text{and} \quad \|\xi\|_{\sigma^m} = \sqrt{\langle \xi, \xi \rangle_{\sigma^m}}$$

the induced norms by the above inner products. Clearly, X_{σ^m} is dense in H_{σ^m} and \hat{H}_{σ^m} , and $H_{\sigma^m} = \hat{H}_{\sigma^m}$ as Hilbert spaces if $m = 1$. Since

$$\begin{aligned} \sum_{j=1}^n \xi_j(t+1) \tilde{e}_j(t+1) &= (\tilde{e}_1(t+1), \dots, \tilde{e}_n(t+1)) (\xi_1(t+1), \dots, \xi_n(t+1))^T \\ &= (\tilde{e}_1(t), \dots, \tilde{e}_n(t)) E_{\sigma^m} E_{\sigma^m} (\xi_1(t), \dots, \xi_n(t))^T \\ &= (\tilde{e}_1(t), \dots, \tilde{e}_n(t)) (\xi_1(t), \dots, \xi_n(t))^T \\ &= \sum_{j=1}^n \xi_j(t) \tilde{e}_j(t) \quad \forall t \in \mathbb{R}, \end{aligned}$$

we obtain Hilbert space isomorphisms

$$\mathfrak{I}_{\sigma^m} : H_{\sigma^m} \rightarrow (T_{\gamma_0^m} \mathcal{H}, \langle \cdot, \cdot \rangle_1) \quad (\text{resp. } \hat{H}_{\sigma^m} \rightarrow (T_{\gamma_0^m} \mathcal{H}, \langle \cdot, \cdot \rangle_m))$$

given by $\mathfrak{I}_{\sigma^m}(\xi) = \sum_{j=1}^n \xi_j \tilde{e}_j$. Under the isomorphisms \mathfrak{I}_{σ} and \mathfrak{I}_{σ^m} the iteration $\tilde{\varphi}_m : T_{\gamma_0} \mathcal{H} \rightarrow T_{\gamma_0^m} \mathcal{H}$ in (5.2) corresponds to the map

$$\tilde{\varphi}_{m\sigma} : H_{\sigma} \rightarrow \hat{H}_{\sigma^m} \quad (5.34)$$

given by $\tilde{\varphi}_{m\sigma}(\xi)(t) = \xi^m(t) = \xi(mt) \forall t \in \mathbb{R}$. So $\tilde{\varphi}_m \circ \mathfrak{I}_{\sigma^m} = \tilde{\varphi}_{m\sigma} \circ \mathfrak{I}_{\sigma}$.

With the exponential map \exp of h we define the C^k map

$$\phi_{\sigma^m} : \mathbb{R} \times B_{2\rho}^n(0) \rightarrow M, \quad (t, v) \mapsto \exp_{\gamma_0^m(t)} \left(\sum_{i=1}^n v_i \tilde{e}_i(t) \right)$$

for some small open ball $B_{2\rho}^n(0) \subset \mathbb{R}^n$. It satisfies

$$\begin{aligned} \phi_{\sigma^m}(t, x) &= \phi_{\sigma}(mt, x), \quad D\phi_{\sigma^m}(t, x)[1, v] = D\phi_{\sigma}(mt, x)[m, v], \\ \phi_{\sigma^m}(t+1, x) &= \phi_{\sigma^m}(t, (E_{\sigma^m} x^T)^T) \quad \text{and} \\ D\phi_{\sigma^m}(t+1, x)[1, v] &= D\phi_{\sigma^m}(t, (E_{\sigma^m} x^T)^T)[1, (E_{\sigma^m} v^T)^T] \end{aligned}$$

for $(t, x) \in \mathbb{R} \times B_{2\rho}^n(0)$ (by shrinking $\rho > 0$ if necessary). This yields a C^k coordinate chart around γ_0^m on \mathcal{H} ,

$$\Phi_{\sigma^m} : \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m}) := \{\xi \in H_{\sigma^m} \mid \|\xi\|_{\sigma^m} < 2\rho\} \rightarrow \mathcal{H}$$

given by $\Phi_{\sigma^m}(\xi)(t) = \phi_{\sigma^m}(t, \xi(t))$ for $\xi \in \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m})$. Note that $d\Phi_{\sigma^m}(0) = \mathfrak{I}_m$.

Let us compute the expression of \mathcal{L}^* in this chart. Define

$$L_{\sigma^m}^*(t, x, v) = L^*(\phi_{\sigma^m}(t, x), D\phi_{\sigma^m}(t, x)[(1, v)]). \quad (5.35)$$

Then for any $t \in \mathbb{R}$ we have $L_{\sigma^m}^*(t, 0, 0) \equiv m^2 c$,

$$\begin{aligned} L_{\sigma^m}^*(t+1, x, v) &= L^*(\phi_{\sigma^m}(t+1, x), D\phi_{\sigma^m}(t+1, x)[(1, v)]) \\ &= L^*(\phi_{\sigma^m}(t, (E_{\sigma^m} x^T)^T), D\phi_{\sigma^m}(t, (E_{\sigma^m} x^T)^T)[1, (E_{\sigma^m} v^T)^T]) \\ &= L_{\sigma^m}^*(t, (E_{\sigma^m} x^T)^T, (E_{\sigma^m} v^T)^T) \end{aligned}$$

and

$$\begin{aligned} D_x L_{\sigma^m}^*(t+1, x, v) &= D_x L_{\sigma^m}^*(t, (E_{\sigma^m} x^T)^T, (E_{\sigma^m} v^T)^T) E_{\sigma^m}, \\ D_v L_{\sigma^m}^*(t+1, x, v) &= D_v L_{\sigma^m}^*(t, (E_{\sigma^m} x^T)^T, (E_{\sigma^m} v^T)^T) E_{\sigma^m}. \end{aligned}$$

It follows from these that

$$\begin{aligned} L_{\sigma^m}^*(t+1, \xi(t+1), \dot{\xi}(t+1)) &= L_{\sigma^m}^*(t, (E_{\sigma^m} \xi(t+1)^T)^T, (E_{\sigma^m} \dot{\xi}(t+1)^T)^T) \\ &= L_{\sigma^m}^*(t, \xi(t), \dot{\xi}(t)) \quad \forall t \in \mathbb{R}, \end{aligned} \quad (5.36)$$

$$D_x L_{\sigma^m}^*(t+1, \xi(t+1), \dot{\xi}(t+1)) = D_x L_{\sigma^m}^*(t, \xi(t), \dot{\xi}(t)) E_{\sigma^m} \quad \forall t \in \mathbb{R}, \quad (5.37)$$

$$D_v L_{\sigma^m}^*(t+1, \xi(t+1), \dot{\xi}(t+1)) = D_v L_{\sigma^m}^*(t, \xi(t), \dot{\xi}(t)) E_{\sigma^m} \quad \forall t \in \mathbb{R} \quad (5.38)$$

for any $\xi \in \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m})$ since $\xi(t+1)^T = E_{\sigma^m} \xi(t)^T$ and $\dot{\xi}(t+1)^T = E_{\sigma^m} \dot{\xi}(t)^T$. Define the action functional $\mathcal{L}_{\sigma^m}^* : \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m}) \rightarrow \mathbb{R}$ by

$$\mathcal{L}_{\sigma^m}^*(\xi) = \int_0^1 L_{\sigma^m}^*(t, \xi(t), \dot{\xi}(t)) dt.$$

Then $\mathcal{L}_{\sigma^m}^* = \mathcal{L}^* \circ \Phi_{\sigma^m}$ on $\mathbf{B}_{2\rho}(\hat{H}_{\sigma^m})$. We claim that it is C^{2-0} . In fact, let $\nabla^m \mathcal{L}_{\sigma^m}^*$ be the gradient of $\mathcal{L}_{\sigma^m}^*$ on \hat{H}_{σ^m} . Then for $\xi \in \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m})$, $\nabla^m \mathcal{L}_{\sigma^m}^*(\xi)$ is given by (B.14) and (B.17). From those we derive

$$\begin{aligned} \frac{d}{dt} \nabla^m \mathcal{L}_{\sigma^m}^*(\xi)(t) &= \frac{e^{mt}}{2} \int_t^\infty e^{-ms} \left(D_x L_{\sigma^m}^*(s, \xi(s), \dot{\xi}(s)) - \mathfrak{G}^{m\xi}(s) \right) ds \\ &\quad - \frac{e^{-mt}}{2} \int_{-\infty}^t e^{ms} \left(D_x L_{\sigma^m}^*(s, \xi(s), \dot{\xi}(s)) - \mathfrak{G}^{m\xi}(s) \right) ds \\ &\quad - \left(\int_0^1 D_v L_{\sigma^m}^*(s, \xi(s), \dot{\xi}(s)) ds \right) \text{diag} \left(\frac{1 + \sigma^m}{2}, 1, \dots, 1 \right) \\ &\quad + D_v L_{\sigma^m}^*(t, \xi(t), \dot{\xi}(t)). \end{aligned} \quad (5.39)$$

Here $\mathfrak{G}^{m\xi}$ is given by (B.17). Let $\hat{A}_{\sigma^m}^*$ be the restriction of the gradient $\nabla^m \mathcal{L}_{\sigma^m}^*$ to $\mathbf{B}_{2\rho}(\hat{H}_{\sigma^m}) \cap X_{\sigma^m}$. Then $\hat{A}_{\sigma^m}^*(\mathbf{B}_{2\rho}(\hat{H}_{\sigma^m}) \cap X_{\sigma^m}) \subset X_{\sigma^m}$. As in the proof of [32, Lemma 3.2] it follows from (B.14), (B.17) and (5.39) that $\hat{A}_{\sigma^m}^*$ is C^1 as a map from

the open subset $\mathbf{B}_{2\rho}(\hat{H}_{\sigma^m}) \cap X_{\sigma^m}$ of X_{σ^m} to X_{σ^m} (and hence the restriction of $\mathcal{L}_{\sigma^m}^*$ to $\mathbf{B}_{2\rho}(\hat{H}_{\sigma^m}) \cap X_{\sigma^m}$, denoted by $\mathcal{L}_{\sigma^m}^{*X}$, is C^2). Since

$$\begin{aligned} d^2 \mathcal{L}_{\sigma^m}^{*X}(\zeta)(\xi, \eta) = \int_0^1 & \left[D_{\tilde{v}\tilde{v}} L_{\sigma^m}^*(t, \zeta(t), \dot{\zeta}(t))(\dot{\xi}(t), \dot{\eta}(t)) \right. \\ & + D_{\tilde{q}\tilde{v}} L_{\sigma^m}^*(t, \gamma(t), \dot{\gamma}(t))(\xi(t), \dot{\eta}(t)) \\ & + D_{\tilde{v}\tilde{q}} L_{\sigma^m}^*(t, \zeta(t), \dot{\zeta}(t))(\dot{\xi}(t), \eta(t)) \\ & \left. + D_{\tilde{q}\tilde{q}} L_{\sigma^m}^*(t, \zeta(t), \dot{\zeta}(t))(\xi(t), \eta(t)) \right] dt \end{aligned}$$

for any $\zeta \in \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m}) \cap X_{\sigma^m}$ and $\xi, \eta \in X_{\sigma^m}$, it is easily checked that the corresponding properties to (i) and (ii) below Lemma 4.2 hold for $\mathcal{L}_{\sigma^m}^{*X}$, that is,

(i) for any $\zeta \in \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m}) \cap X_{\sigma^m}$ there exists a constant $C(\zeta)$ such that

$$|d^2 \mathcal{L}_{\sigma^m}^{*X}(\zeta)(\xi, \eta)| \leq C(\zeta) \|\xi\|_{\sigma^m} \cdot \|\eta\|_{\sigma^m} \quad \forall \xi, \eta \in X_{\sigma^m};$$

(ii) $\forall \varepsilon > 0, \exists \delta_0 > 0$, such that for all $\zeta_1, \zeta_2 \in \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m}) \cap X_{\sigma^m}$ with $\|\zeta_1 - \zeta_2\|_{C^1} < \delta_0$,

$$|d^2 \mathcal{L}_{\sigma^m}^{*X}(\zeta_1)(\xi, \eta) - d^2 \mathcal{L}_{\sigma^m}^{*X}(\zeta_2)(\xi, \eta)| \leq \varepsilon \|\xi\|_{\sigma^m} \cdot \|\eta\|_{\sigma^m} \quad \forall \xi, \eta \in X_{\sigma^m}.$$

It follows that there exists a continuous map $\hat{B}_{\sigma^m}^* : \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m}) \cap X_{\sigma^m} \rightarrow L_s(\hat{H}_{\sigma^m})$ with respect to the induced topology on $\mathbf{B}_{2\rho}(\hat{H}_{\sigma^m}) \cap X_{\sigma^m}$ by X_{σ^m} , such that

$$\langle d\hat{A}_{\sigma^m}^*(\zeta)\xi, \eta \rangle_{\sigma^m} = d^2 \mathcal{L}_{\sigma^m}^{*X}(\zeta)(\xi, \eta) = \langle \hat{B}_{\sigma^m}^*(\zeta)\xi, \eta \rangle_{\sigma^m}$$

for any $\zeta \in \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m}) \cap X_{\sigma^m}$ and $\xi, \eta \in X_{\sigma^m}$. These and similar arguments to those of [32] lead to

Claim 5.9 $(\hat{H}_{\sigma^m}, X_{\sigma^m}, \mathcal{L}_{\sigma^m}^*, \hat{A}_{\sigma^m}^*, \hat{B}_{\sigma^m}^*)$ satisfy the conditions of Theorem A.1 around the critical point $0 \in \hat{H}_{\sigma^m}$.

For $m = 1$ let us write $\sigma^m = \sigma$ below. Observe that $\tilde{\mathcal{L}}_\sigma^*$ has one-dimensional critical manifold $S := \Phi_\sigma^{-1}(S^1 \cdot \gamma_0 \cap \text{Im}(\Phi_\sigma))$, and that $T_{\gamma_0}(S^1 \cdot \gamma_0) = \dot{\gamma}_0 \mathbb{R} \subset W^{1,2}(\gamma_0^* TM)$. Since $d\Phi_\sigma(0) = \mathfrak{I}_\sigma$ is an isomorphism there exists a unique $\zeta_0 \in H_\sigma = \hat{H}_\sigma$ satisfying $d\Phi_\sigma(0)(\zeta_0) = \dot{\gamma}_0$. That is, for any $t \in \mathbb{R}$,

$$\dot{\gamma}_0(t) = \frac{d}{ds} \Big|_{s=0} \Phi_\sigma(s\zeta_0)(t) = \frac{d}{ds} \Big|_{s=0} \exp_{\gamma_0(t)} \left(\sum_{k=1}^n s\zeta_{0k}(t)e_k(t) \right) = \sum_{k=1}^n \zeta_{0k}(t)e_k(t).$$

Hence $h(\dot{\gamma}_0(t), e_j(t)) = \zeta_{0j}(t)$ for any $t \in \mathbb{R}$ and $j = 1, \dots, n$. Clearly, $T_0 S = \zeta_0 \mathbb{R}$. The normal space of S at $0 \in S$ is the orthogonal complementary of $\zeta_0 \mathbb{R}$ in the Hilbert space $\hat{H}_\sigma = H_\sigma$, denoted by $H_{\sigma,0}$. Note that $\zeta_0 \in X_\sigma$ and that

$$\zeta_0^m = \tilde{\varphi}_{m\sigma}(\zeta_0) \in X_{\sigma^m} \quad \text{and} \quad \mathfrak{I}_{\sigma^m}(\zeta_0^m) = (\dot{\gamma}_0)^m. \quad (5.40)$$

Let $\hat{H}_{\sigma^m,0}$ be the orthogonal complementary of $\zeta_0^m \mathbb{R}$ in \hat{H}_{σ^m} , and let $\hat{X}_{\sigma^m,0} := \hat{H}_{\sigma^m,0} \cap X_{\sigma^m}$. Then $\tilde{\varphi}_{m\sigma}$ also maps $X_{\sigma,0} = H_{\sigma,0} \cap X_\sigma$ into $\hat{X}_{\sigma^m,0} := \hat{H}_{\sigma^m,0} \cap X_{\sigma^m}$ by (5.40). Denote by $\hat{\mathcal{L}}_{\sigma^m,0}^*$ the restriction of $\mathcal{L}_{\sigma^m}^*$ to $\mathbf{B}_{2\rho}(\hat{H}_{\sigma^m,0})$. Then

$$\nabla^m \hat{\mathcal{L}}_{\sigma^m,0}^*(\xi) = \nabla^m \tilde{\mathcal{L}}_{\sigma^m}^*(\xi) - \frac{\langle \nabla^m \tilde{\mathcal{L}}_{\sigma^m}^*(\xi), \zeta_0^m \rangle_{\sigma^m}}{\|\zeta_0^m\|_{\sigma^m}^2} \zeta_0^m \quad (5.41)$$

for $\xi \in \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m,0})$. It follows that the restriction of $\nabla^m \hat{\mathcal{L}}_{\sigma^m,0}$ to $\mathbf{B}_{2\rho}(\hat{H}_{\sigma^m,0}) \cap \hat{X}_{\sigma^m,0}$, denoted by $\hat{A}_{\sigma^m,0}^*$, is a C^1 -map into $\hat{X}_{\sigma^m,0}$, and

$$\hat{A}_{\sigma^m,0}^*(\xi) = \hat{A}_{\sigma^m}^*(\xi) - \frac{\langle \hat{A}_{\sigma^m}^*(\xi), \zeta_0^m \rangle_{\sigma^m}}{\|\zeta_0^m\|_{\sigma^m}^2} \zeta_0^m \quad (5.42)$$

for $\xi \in \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m,0}) \cap \hat{X}_{\sigma^m,0}$. Let $\hat{\mathcal{L}}_{\sigma^m,0}^{*X}$ be the restriction of $\hat{\mathcal{L}}_{\sigma^m}^{*X}$ to $\mathbf{B}_{2\rho}(\hat{H}_{\sigma^m,0}) \cap \hat{X}_{\sigma^m,0}$, and let $\hat{B}_{\sigma^m,0}^*(\xi)$ be the extension of the continuous symmetric bilinear form $d^2(\hat{\mathcal{L}}_{\sigma^m,0}^{*X})(\xi)$ on $\hat{H}_{\sigma^m,0}$ for $\xi \in \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m,0}) \cap \hat{X}_{\sigma^m,0}$. Then

$$\hat{B}_{\sigma^m,0}^*(\xi)\eta = \hat{B}_{\sigma^m}^*(\xi)\eta - \frac{\langle \hat{B}_{\sigma^m}^*(\xi)\eta, \zeta_0^m \rangle_{\sigma^m}}{\|\zeta_0^m\|_{\sigma^m}^2} \zeta_0^m \quad \forall \eta \in \hat{H}_{\sigma^m,0}.$$

It is easy to see that $\hat{B}_{\sigma^m,0}^* : \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m,0}) \cap \hat{X}_{\sigma^m,0} \rightarrow L_s(\hat{H}_{\sigma^m,0})$ is continuous with respect to the induced topology on $\mathbf{B}_{2\rho}(\hat{H}_{\sigma^m,0}) \cap \hat{X}_{\sigma^m,0}$ by $\hat{X}_{\sigma^m,0}$, and

$$\langle d\hat{A}_{\sigma^m,0}^*(\xi)\zeta, \eta \rangle_{\sigma^m} = d^2(\hat{\mathcal{L}}_{\sigma^m,0}^{*X})(\xi)(\zeta, \eta) = \langle \hat{B}_{\sigma^m,0}^*(\xi)\zeta, \eta \rangle_{\sigma^m}$$

for any $\xi \in \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m,0}) \cap \hat{X}_{\sigma^m,0}$ and $\zeta, \eta \in \hat{X}_{\sigma^m,0}$. By Claim 5.9 and Theorem A.3 we obtain

Claim 5.10 $(\hat{H}_{\sigma^m,0}, \hat{X}_{\sigma^m,0}, \hat{\mathcal{L}}_{\sigma^m,0}^*, \hat{A}_{\sigma^m,0}^*, \hat{B}_{\sigma^m,0}^*)$ satisfy the conditions of Theorem A.1 around the critical point $0 \in \hat{H}_{\sigma^m,0}$.

Remark 5.11 From the arguments above Claim 5.9 one easily sees that the conditions of Theorem A.5 are satisfied for $(\hat{H}_{\sigma^m}, X_{\sigma^m}, \mathcal{L}_{\sigma^m}^*, \hat{A}_{\sigma^m}^*, \hat{B}_{\sigma^m}^*)$ around the critical point 0. By Theorem A.7 $(\hat{H}_{\sigma^m,0}, \hat{X}_{\sigma^m,0}, \hat{\mathcal{L}}_{\sigma^m,0}^*, \hat{A}_{\sigma^m,0}^*, \hat{B}_{\sigma^m,0}^*)$ satisfies the conditions of Theorem A.5 around the critical point 0 too.

As in (3.17) let $\Psi_{\gamma_0^m}$ denote the restriction of $\text{EXP}|_{N\varphi_m(\mathcal{O})(\delta)}$ to $N\varphi_m(\mathcal{O})_{\gamma_0^m}(\delta)$. Assume $\delta < 2\rho$. For $\xi \in \mathbf{B}_\delta(\hat{H}_{\sigma^m})$, by the arguments above (5.35) we get

$$\begin{aligned} \Phi_{\sigma^m}(\xi)(t) &= \phi_{\sigma^m}(t, \xi(t)) = \exp_{\gamma_0^m(t)} \left(\sum_{i=1}^n \xi_i(t) \tilde{e}_i(t) \right) \\ &= \exp_{\gamma_0^m(t)} (\mathcal{J}_{\sigma^m}(\xi)(t)) = \exp_{\gamma_0^m(t)} ((d\Phi_{\sigma^m}(0)\xi)(t)) \\ &= \Psi_{\gamma_0^m}(d\Phi_{\sigma^m}(0)\xi)(t). \end{aligned} \quad (5.43)$$

That is, $\Psi_{\gamma_0^m} \circ d\Phi_{\sigma^m}(0) = \Phi_{\sigma^m}$ on $\mathbf{B}_\delta(\hat{H}_{\sigma^m})$. Recall that $\mathcal{L}_{\sigma^m}^* = \mathcal{L}^* \circ \Phi_{\sigma^m}$ on $\mathbf{B}_{2\rho}(\hat{H}_{\sigma^m})$ and $\hat{\mathcal{F}}_{\gamma_0^m}^* = \mathcal{L}^* \circ \Psi_{\gamma_0^m}$ by (5.4). These and (5.43) lead to

$$\hat{\mathcal{F}}_{\gamma_0^m}^* \circ d\Phi_{\sigma^m}(0) = \mathcal{L}^* \circ \Psi_{\gamma_0^m} \circ d\Phi_{\sigma^m}(0) = \mathcal{L}^* \circ \Phi_{\sigma^m} = \mathcal{L}_{\sigma^m}^*$$

on $\mathbf{B}_\delta(\hat{H}_{\sigma^m})$, and hence

$$\hat{\mathcal{F}}_{\gamma_0^m}^* \circ d\Phi_{\sigma^m}(0) = \hat{\mathcal{L}}_{\sigma^m,0}^* \quad \text{on} \quad \mathbf{B}_\delta(\hat{H}_{\sigma^m,0}). \quad (5.44)$$

Since $\mathcal{I}_{\sigma^m} = d\Phi_{\sigma^m}(0)$ restricts to a Hilbert space isomorphism $\mathcal{I}_{\sigma^m,0}$ from $\hat{H}_{\sigma^m,0}$ to $\hat{N}\varphi_m(\mathcal{O})_{\gamma_0^m}$ and a Banach space isomorphism $\mathcal{I}_{\sigma^m,0}^X$ from $\hat{X}_{\sigma^m,0}$ to $X\hat{N}\varphi_m(\mathcal{O})_{\gamma_0^m}$, by (5.44) ones easily prove $\hat{A}_{\gamma_0^m}^* = \mathcal{I}_{\sigma^m,0}^X \circ \hat{A}_{\sigma^m,0}^* \circ (\mathcal{I}_{\sigma^m,0}^X)^{-1}$ and

$$\hat{\mathbf{B}}_{\gamma_0^m}^*(\xi) = \mathcal{I}_{\sigma^m,0} \circ \hat{B}_{\sigma^m,0}^* ((\mathcal{I}_{\sigma^m,0}^X)^{-1}\xi) \circ (\mathcal{I}_{\sigma^m,0})^{-1} \quad \forall \xi \in \mathbf{B}_\delta(\hat{H}_{\sigma^m,0}) \cap \hat{X}_{\sigma^m,0}.$$

By Claim 5.10 and Theorem A.4, $(\hat{N}\varphi_m(\mathcal{O})_{\gamma_0^m}, X\hat{N}\varphi_m(\mathcal{O})_{\gamma_0^m}, \hat{\mathcal{F}}_{\gamma_0^m}^*, \hat{A}_{\gamma_0^m}^*, \hat{B}_{\gamma_0^m}^*)$ satisfies the conditions of Theorem A.1 (and hence Theorem A.5) around the critical point $\gamma_0^m \equiv 0_{\gamma_0^m}$. Proposition 5.4 is proved. \square

5.4 Proof of Theorem 1.8

Step 1. Proving $p > 0$. By a contradiction argument we assume $p = 0$.

Case 1. $m^0(S^1 \cdot \gamma_0) = 0$. Since $m^-(\gamma_0) = 0$, by Theorem 1.7 we have $C_q(\mathcal{L}, \mathcal{O}; \mathbb{K}) = H_q(S^1; \mathbb{K})$ and hence $C_q(\mathcal{L}, \mathcal{O}; \mathbb{K}) = 0$ for $q \notin \{0, 1\}$. By the assumption p must be larger than zero.

Case 2. $m^0(S^1 \cdot \gamma_0) > 0$. Since $m^-(\gamma_0) = 0$ we derive from Theorem 1.7 that

$$\begin{aligned} C_0(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) &= (C_0(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1} \quad \forall x \in S^1 \cdot \gamma_0, \\ C_1(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) &= (C_0(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1} \oplus (C_1(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1} \quad \forall x \in S^1 \cdot \gamma_0. \end{aligned}$$

Hence p must be more than zero too.

Step 2. Proving (i). If $m^0(S^1 \cdot \gamma_0) = 0$, then $\mathcal{L}^* \circ \text{EXP} \circ \Upsilon(u) = \|u\|_1^2$ for any $u \in N\mathcal{O}(\epsilon)$ by Theorem 1.5(iii). Moreover $\mathcal{L}^* \leq \mathcal{L}$ and $\mathcal{L}^* = \mathcal{L}$ on $S^1 \cdot \gamma_0$. The claim follows directly.

If $m^0(S^1 \cdot \gamma_0) > 0$, by Theorem 1.7 for any $x \in S^1 \cdot \gamma_0$ we have

$$\begin{aligned} 0 \neq C_1(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) &= (C_0(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1} \oplus (C_1(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1}, \\ 0 = C_2(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) &= (C_1(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1} \oplus (C_2(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1}. \end{aligned}$$

It follows that $C_0(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}) \neq 0$. By Theorem 4.6 on the page 43 of [16] 0_x must be a local minimum of $\mathcal{L}_{\Delta x}^\circ$. This and Theorem 1.5(iii) imply that x is a local minimum of \mathcal{L}^* and hence of \mathcal{L} because $\mathcal{L}^* \leq \mathcal{L}$ and $\mathcal{L}^* = \mathcal{L}$ on $S^1 \cdot \gamma_0$ as above.

Step 3. Proving (ii). If $m^0(S^1 \cdot \gamma_0) = 0$, by the proof of Case 1 in Step 1 we have

$$C_p(\mathcal{L}, \mathcal{O}; \mathbb{K}) = H_p(S^1; \mathbb{K}) = 0 \quad \text{since } p \geq 2.$$

This case cannot occur. Hence it must hold that $m^0(S^1 \cdot \gamma_0) > 0$. By Theorem 1.7

$$\begin{aligned} 0 \neq C_p(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) &= (C_{p-1}(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1} \oplus (C_p(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1}, \\ 0 = C_{p+1}(\mathcal{L}, S^1 \cdot \gamma_0; \mathbb{K}) &= (C_p(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1} \oplus (C_{p+1}(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}))^{S_x^1} \end{aligned}$$

for any $x \in S^1 \cdot \gamma_0$. It follows that $C_{p-1}(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}) \neq 0$. By Example 1 on the page 33 of [16] 0_x may not be a local minimum of $\mathcal{L}_{\Delta x}^\circ$. This and Theorem 1.6 imply that x is not a local minimum of $\mathcal{L}|_{\mathcal{X}}$ and thus of \mathcal{L} . \square

6 Critical groups of iterated closed geodesics

In this section we always assume that $\mathcal{O} = S^1 \cdot \gamma_0$ and $\varphi_m(\mathcal{O}) = S^1 \cdot \gamma_0^m$ are two isolated critical orbits of \mathcal{L} in \mathcal{H} . Theorem 1.9 is equivalent to the following theorem.

Theorem 6.1 *Let \mathbb{K} be a field. Suppose that $m^-(\mathcal{O}) = m^-(\varphi_m(\mathcal{O}))$ and $m^0(\mathcal{O}) = m^0(\varphi_m(\mathcal{O}))$. Then there exist S^1 -invariant neighborhoods \mathcal{U}_i of $\varphi_i(\mathcal{O})$, $i = 1, m$, $\varphi_m(\mathcal{U}_1) \subset \mathcal{U}_m$, such that φ_m induces isomorphisms*

$$(\varphi_m)_* : H_*(\mathcal{L}_c \cap \mathcal{U}_1, \mathcal{L}_c \cap (\mathcal{U}_1 \setminus \{\mathcal{O}\}); \mathbb{K}) \rightarrow H_*(\mathcal{L}_{m^2c} \cap \mathcal{U}_m, \mathcal{L}_{m^2c} \cap (\mathcal{U}_m \setminus \{\varphi_m(\mathcal{O})\}); \mathbb{K}).$$

Here $c = \mathcal{L}|_{\mathcal{O}}$ and hence $m^2c = \mathcal{L}|_{\varphi_m(\mathcal{O})}$ by (5.1). (In particular, φ_m induces isomorphisms from $C_*(\mathcal{L}, \mathcal{O}; \mathbb{K})$ to $C_*(\mathcal{L}, \varphi_m(\mathcal{O}); \mathbb{K})$.)

As in the proof of [32, Theorem 5.1] this is also equivalent to the following

Theorem 6.2 *Let $c = \mathcal{L}(\gamma)$ and take a small $\varepsilon > 0$ such that $c \pm \varepsilon$ and $m^2c \pm m^2\varepsilon$ are all regular values of \mathcal{L} . Then there exist topological Gromoll-Meyer pairs of \mathcal{L} at \mathcal{O} and $\varphi_m(\mathcal{O})$, (W_1, W_1^-) and (W_2, W_2^-) , such that*

$$\begin{aligned} (W_1, W_1^-) &\subset (\mathcal{L}^{-1}[c - \varepsilon, c + \varepsilon], \mathcal{L}^{-1}(c - \varepsilon)), \\ (W_2, W_2^-) &\subset (\mathcal{L}^{-1}[m^2c - m^2\varepsilon, m^2c + m^2\varepsilon], \mathcal{L}^{-1}(m^2c - m^2\varepsilon)), \\ (\varphi_m(W_1), \varphi_m(W_1^-)) &\subset (W_2, W_2^-) \end{aligned}$$

and that φ_m induces isomorphisms

$$(\varphi_m)_* : H_*(W_1, W_1^-; \mathbb{K}) \rightarrow H_*(W_2, W_2^-; \mathbb{K}).$$

Here the so-called topological Gromoll-Meyer pair we mean the homeomorphism image of a Gromoll-Meyer pair.

Theorem 1.10 is contained in the following result.

Theorem 6.3 *Let \mathbb{K} be a field. Suppose that $m^0(\mathcal{O}) = m^0(\varphi_m(\mathcal{O}))$. Then for any $x \in \mathcal{O}$ it holds that*

$$\dim C_*(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K}) = \dim C_*(\hat{\mathcal{L}}_{\Delta x^m}^\circ, 0; \mathbb{K}) = \dim C_*({}^m\mathcal{L}_{\Delta x^m}^\circ, 0; \mathbb{K}). \quad (6.1)$$

If the characteristic of \mathbb{K} is zero or prime to orders of $S_{\gamma_0}^1$ and $S_{\gamma_0^m}^1$ then

$$\dim C_*(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K})^{S_x^1} = \dim C_*(\hat{\mathcal{L}}_{\Delta x^m}^\circ, 0; \mathbb{K})^{S_{x^m}^1}, \quad (6.2)$$

$$\dim C_*(\mathcal{L}_{\Delta x}^\circ, 0; \mathbb{K})^{S_x^1} = \dim C_*({}^m\mathcal{L}_{\Delta x^m}^\circ, 0; \mathbb{K})^{S_{x^m}^1}. \quad (6.3)$$

6.1 Proof of Theorem 6.1

Comparing with the case considered in Riemannian geometry as in [25] there exist the following two problems which form obstructions for using the methods of [25]:

- So far one has not proved a splitting lemma around \mathcal{O} for the functional \mathcal{L} on the Hilbert manifold $\mathcal{H} = \Lambda M$;
- There exists a splitting lemma around \mathcal{O} for the restriction $\mathcal{L}|_{\mathcal{X}}$ of the functional \mathcal{L} to the Banach manifold $\mathcal{X} = C^1(S^1, M)$, but $\mathcal{L}|_{\mathcal{X}}$ does not satisfy the (PS) condition and hence has no so-called Gromoll-Meyer pairs.

Our Theorem 5.1(v) will help us overcome these difficulties. Follow the notations in Section 5.1. Note that $\tilde{\varphi}_m \circ \text{EXP} = \text{EXP} \circ \tilde{\varphi}_m$ and that $\tilde{\varphi}_m(N\mathcal{O}(\delta)) \subset \hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\delta)$. Clearly, Theorem 6.1 is equivalent to

Proposition 6.4 *Under the assumptions of Theorem 6.1 there exist S^1 -invariant neighborhoods \mathcal{U}_1 of \mathcal{O} in $N\mathcal{O}(\delta)$, and \mathcal{U}_m of $\varphi_m(\mathcal{O})$ in $\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\delta)$, such that $\tilde{\varphi}_m(\mathcal{U}_1) \subset \mathcal{U}_m$ and that $\tilde{\varphi}_m$ induces isomorphisms*

$$(\tilde{\varphi}_m)_* : H_*\left(\mathcal{F}_c \cap \mathcal{U}_1, \mathcal{F}_c \cap (\mathcal{U}_1 \setminus \{\mathcal{O}\}); \mathbb{K}\right) \rightarrow H_*\left(\hat{\mathcal{F}}_{m^2c} \cap \mathcal{U}_m, \hat{\mathcal{F}}_{m^2c} \cap (\mathcal{U}_m \setminus \{\varphi_m(\mathcal{O})\}); \mathbb{K}\right).$$

Write $\mathcal{U}_1^X = \mathcal{U}_1 \cap XN\mathcal{O}$ (resp. $\mathcal{U}_m^X = \mathcal{U}_m \cap X\hat{N}\varphi_m(\mathcal{O})$) as an open subset of $XN\mathcal{O}$ (resp. $X\hat{N}\varphi_m(\mathcal{O})$). By Theorem 5.1(v) the inclusions

$$((\mathcal{F}^X)_c \cap \mathcal{U}_1^X, (\mathcal{F}^X)_c \cap \mathcal{U}_1^X \setminus \mathcal{O}) \hookrightarrow (\mathcal{F}_c \cap \mathcal{U}_1, \mathcal{F}_c \cap \mathcal{U} \setminus \mathcal{O})$$

and

$$((\hat{\mathcal{F}}^X)_{m^2c} \cap \mathcal{U}_m^X, (\hat{\mathcal{F}}^X)_{m^2c} \cap \mathcal{U}_m^X \setminus \varphi_m(\mathcal{O})) \hookrightarrow (\hat{\mathcal{F}}_{m^2c} \cap \mathcal{U}_m, \hat{\mathcal{F}}_{m^2c} \cap \mathcal{U}_m \setminus \varphi_m(\mathcal{O})),$$

induce isomorphisms among their relative homology groups. Moreover these two inclusions commute with $\tilde{\varphi}_m$. Hence Proposition 6.4 is equivalent to

Proposition 6.5 *$\tilde{\varphi}_m$ induces isomorphisms*

$$(\tilde{\varphi}_m)_* : H_*\left((\mathcal{F}^X)_c \cap \mathcal{U}_1^X, (\mathcal{F}^X)_c \cap \mathcal{U}_1^X \setminus \mathcal{O}; \mathbb{K}\right) \rightarrow H_*\left((\hat{\mathcal{F}}^X)_{m^2c} \cap \mathcal{U}_m^X, (\hat{\mathcal{F}}^X)_{m^2c} \cap \mathcal{U}_m^X \setminus \varphi_m(\mathcal{O}); \mathbb{K}\right).$$

Take a S^1 -invariant open neighborhood \hat{V}_1 (resp. \hat{V}_m) of \mathcal{O} (resp. $\varphi_m(\mathcal{O})$) in $XN\mathcal{O}$ (resp. $X\hat{N}\varphi_m(\mathcal{O})$) such that $\hat{V}_1 \subset \mathcal{U}_1^X$, $\hat{V}_m \subset \mathcal{U}_m^X$ and $\tilde{\varphi}_m(\hat{V}_1) \subset \hat{V}_m$. By the excision property of relative homology groups Proposition 6.5 is equivalent to

Proposition 6.6 *$\tilde{\varphi}_m$ induces isomorphisms*

$$(\tilde{\varphi}_m)_* : H_*\left((\mathcal{F}^X)_c \cap \hat{V}_1, (\mathcal{F}^X)_c \cap \hat{V}_1 \setminus \mathcal{O}; \mathbb{K}\right) \rightarrow H_*\left((\hat{\mathcal{F}}^X)_{m^2c} \cap \hat{V}_m, (\hat{\mathcal{F}}^X)_{m^2c} \cap \hat{V}_m \setminus \varphi_m(\mathcal{O}); \mathbb{K}\right).$$

Assume that Theorems 1.6, 5.2 take the same ϵ by shrinking. We need to study the relationships between V , \mathfrak{h} , Ψ in Theorem 1.6 and \widehat{V} , $\widehat{\mathfrak{h}}$, $\widehat{\Psi}$ in Theorem 5.2. It suffices to restrict to the case of a fibre.

Let $\mathcal{L}_{\sigma,0}$ and $\hat{\mathcal{L}}_{\sigma^m,0}$ be the restrictions of \mathcal{L}_σ and \mathcal{L}_{σ^m} to $\mathbf{B}_{2\delta}(H_{\sigma,0})$ and $\mathbf{B}_{2\sqrt{m}\delta}(\hat{H}_{\sigma^m,0})$, respectively, where

$$\begin{aligned}\mathcal{L}_\sigma(\xi) &= \int_0^1 L_\sigma(t, \xi(t), \dot{\xi}(t)) dt \quad \forall \xi \in \mathbf{B}_{2\delta}(H_\sigma), \\ \mathcal{L}_{\sigma^m}(\xi) &= \int_0^1 L_{\sigma^m}(t, \xi(t), \dot{\xi}(t)) dt \quad \forall \xi \in \mathbf{B}_{2\sqrt{m}\delta}(\hat{H}_{\sigma^m}),\end{aligned}$$

and

$$\begin{aligned}L_\sigma(t, x, v) &= L(\phi(t, x), D\phi(t, x)[(1, v)]), \\ L_{\sigma^m}(t, x, v) &= L(\phi_m(t, x), D\phi_m(t, x)[(1, v)])\end{aligned}$$

as in (5.35). With the same deduction as that of (5.44) we may obtain

$$\mathcal{F}_{\gamma_0} \circ d\Phi_\sigma(0) = \mathcal{L}_{\sigma,0} \quad \text{on} \quad \mathbf{B}_\delta(\hat{H}_{\sigma,0}), \quad (6.4)$$

$$\hat{\mathcal{F}}_{\gamma_0^m} \circ d\Phi_{\sigma^m}(0) = \hat{\mathcal{L}}_{\sigma^m,0} \quad \text{on} \quad \mathbf{B}_{\sqrt{m}\delta}(\hat{H}_{\sigma^m,0}), \quad (6.5)$$

by shrinking $\delta > 0$. Note that

$$\begin{aligned}L_{\sigma^m}(t, x, v) &= L(\phi_{\sigma^m}(t, x), D\phi_{\sigma^m}(t, x)[(1, v)]) \\ &= L(\phi(mt, x), D\phi(mt, x)[(m, v)]) \\ &= L(\phi(mt, x), mD\phi(mt, x)[(1, v/m)]) \\ &= m^2 L(\phi(mt, x), D\phi(mt, x)[(1, v/m)]) \\ &= m^2 L_\sigma(mt, x, v/m).\end{aligned} \quad (6.6)$$

(Here we use the homogeneity of F !) Let $\tilde{\varphi}_{m\sigma} : H_\sigma \rightarrow \hat{H}_{\sigma^m}$ be as in (5.34). It is a Hilbert space embedding up to a factor m^2 by (5.33), that is,

$$\langle \tilde{\varphi}_{m\sigma}(\xi), \tilde{\varphi}_{m\sigma}(\eta) \rangle_{\sigma^m} = m^2 \langle \xi, \eta \rangle_{1,\sigma} \quad \forall \xi, \eta \in H_\sigma.$$

Since $\xi^m(t + 1/m) = \xi^m(t)E_\sigma$ for $\xi^m \in \tilde{\varphi}_{m\sigma}(\mathbf{B}_{2\delta}(H_\sigma)) \subset \mathbf{B}_{2\sqrt{m}\delta}(\hat{H}_{\sigma^m})$ it is easily checked that for all $t \in \mathbb{R}$,

$$L_{\sigma^m}(t + \frac{1}{m}, \xi^m(t + \frac{1}{m}), (\xi^m)^\cdot(t + \frac{1}{m})) = L_{\sigma^m}(t, \xi^m(t), (\xi^m)^\cdot(t)), \quad (6.7)$$

$$\begin{aligned}D_x L_{\sigma^m}(t + \frac{1}{m}, \xi^m(t + \frac{1}{m}), (\xi^m)^\cdot(t + \frac{1}{m})) \\ = D_x L_{\sigma^m}(t, \xi^m(t), (\xi^m)^\cdot(t)) E_\sigma,\end{aligned} \quad (6.8)$$

$$\begin{aligned}D_v L_{\sigma^m}(t + \frac{1}{m}, \xi^m(t + \frac{1}{m}), (\xi^m)^\cdot(t + \frac{1}{m})) \\ = D_v L_{\sigma^m}(t, \xi^m(t), (\xi^m)^\cdot(t)) E_\sigma.\end{aligned} \quad (6.9)$$

Since (5.36)-(5.38) also hold when $L_{\sigma^m}^*$ is replaced by L_{σ^m} , for $\xi^m = \tilde{\varphi}_{m\sigma}(\xi) \in \tilde{\varphi}_{m\sigma}(\mathbf{B}_{2\delta}(H_\sigma))$, by (6.7) and (6.6) we get

$$\begin{aligned}\mathcal{L}_{\sigma^m}(\xi^m) &= m \int_0^{1/m} L_{\sigma^m}(t, \xi^m(t), (\xi^m)'(t)) dt \\ &= m \int_0^{1/m} L_{\sigma^m}(t, \xi(mt), m\dot{\xi}(mt)) dt \\ &= m^3 \int_0^{1/m} L_{\sigma}(mt, \xi(mt), \dot{\xi}(mt)) dt \\ &= m^2 \mathcal{L}_{\sigma}(\xi).\end{aligned}\tag{6.10}$$

As in (5.41) the gradient of $\hat{\mathcal{L}}_{\sigma^m,0}$ at $\xi \in \mathbf{B}_{2\sqrt{m}\delta}(\hat{H}_{\sigma^m,0})$ is given by

$$\nabla^m \hat{\mathcal{L}}_{\sigma^m,0}(\xi) = \nabla^m \mathcal{L}_{\sigma^m}(\xi) - \frac{\langle \nabla^m \mathcal{L}_{\sigma^m}(\xi), \zeta_0^m \rangle_{\sigma^m}}{\|\zeta_0^m\|_{\sigma^m}^2} \zeta_0^m.$$

In particular, for $\xi \in \mathbf{B}_{2\delta}(H_{\sigma,0})$ we have

$$\nabla \mathcal{L}_{\sigma,0}(\xi) = \nabla \mathcal{L}_{\sigma}(\xi) - \frac{\langle \nabla \mathcal{L}_{\sigma}(\xi), \zeta_0 \rangle_{1,\sigma}}{\|\zeta_0\|_{1,\sigma}^2} \zeta_0.$$

By Claim B.3, for $\xi \in \mathbf{B}_{2\delta}(H_\sigma)$ we get

$$\frac{\langle \nabla^m \mathcal{L}_{\sigma^m}(\xi^m), \zeta_0^m \rangle_{\sigma^m}}{\|\zeta_0^m\|_{\sigma^m}^2} \zeta_0^m = \frac{\langle \tilde{\varphi}_{m\sigma}(\nabla \mathcal{L}_{\sigma,0}(\xi)), \zeta_0^m \rangle_{\sigma^m}}{\|\zeta_0^m\|_{\sigma^m}^2} \zeta_0^m = \tilde{\varphi}_{m\sigma} \left(\frac{\langle \nabla \mathcal{L}_{\sigma}(\xi), \zeta_0 \rangle_{1,\sigma}}{\|\zeta_0\|_{1,\sigma}^2} \zeta_0 \right)$$

and hence

$$\begin{aligned}\nabla^m \hat{\mathcal{L}}_{\sigma^m,0}(\xi^m) &= \tilde{\varphi}_{m\sigma}(\nabla \mathcal{L}_{\sigma,0}(\xi)) \quad \forall \xi \in \mathbf{B}_{2\delta}(H_{\sigma,0}), \\ \hat{A}_{\sigma^m,0}(\xi^m) &= \tilde{\varphi}_{m\sigma}(A_{\sigma,0}(\xi)) \quad \forall \xi \in \mathbf{B}_{2\delta}(H_{\sigma,0}) \cap X_{\sigma,0}.\end{aligned}$$

From these, (6.4)-(6.5) and (6.10) it follows that

$$\hat{\mathcal{F}}_{\gamma_0^m}(\tilde{\varphi}_m(\xi)) = m^2 \mathcal{F}_{\gamma_0}(\xi), \quad \forall \xi \in N\mathcal{O}(2\delta)_{\gamma_0} \tag{6.11}$$

$$\hat{\nabla} \hat{\mathcal{F}}_{\gamma_0^m}(\tilde{\varphi}_m(\xi)) = \tilde{\varphi}_m(\nabla \mathcal{F}_{\gamma_0}(\xi)) \quad \forall \xi \in N\mathcal{O}(2\delta)_{\gamma_0}, \tag{6.12}$$

$$\hat{A}_{\gamma_0^m}(\tilde{\varphi}_m(\xi)) = \tilde{\varphi}_m(A_{\gamma_0}(\xi)) \quad \forall \xi \in N\mathcal{O}(2\delta)_{\gamma_0} \cap XN\mathcal{O}_{\gamma_0}, \tag{6.13}$$

$$\hat{B}_{\gamma_0^m} \circ \tilde{\varphi}_m = \tilde{\varphi}_m \circ B_{\gamma_0}. \tag{6.14}$$

Recall that $\mathcal{F}_{\gamma_0}^X$ is the restriction of $\mathcal{F}_{\gamma_0} = \mathcal{L} \circ \text{EXP}_{\gamma_0}|_{N(\mathcal{O})(\delta)_{\gamma_0}}$ to $T_{\gamma_0}\mathcal{X}$. Let

$$\begin{aligned}\mathcal{F}_{\gamma_0}^X \circ \Psi_{\gamma_0}(v) &= \frac{1}{2} B_{\gamma_0}(\mathbf{P}_{\gamma_0}^+ v, \mathbf{P}_{\gamma_0}^+ v) - \|\mathbf{P}_{\gamma_0}^- v\|_1^2 + \mathcal{L}_{\Delta\gamma_0}^\circ(\mathbf{P}_{\gamma_0}^0 v) \\ &\equiv \beta_{\gamma_0}(v) + \alpha_{\gamma_0}(v) \quad \forall v \in XN\mathcal{O}(\epsilon)_{\gamma_0}\end{aligned}$$

as in Theorem 1.6, where $\alpha_{\gamma_0}(v) = \mathcal{L}_{\Delta\gamma_0}^\circ(\mathbf{P}_{\gamma_0}^0 v)$, and let

$$\begin{aligned}\hat{\mathcal{F}}_{\gamma_0^m}^X \circ \hat{\Psi}_{\gamma_0^m}(v) &= \frac{1}{2} \hat{B}_{\gamma_0^m}(\hat{\mathbf{P}}_{\gamma_0^m}^+ v, \hat{\mathbf{P}}_{\gamma_0^m}^+ v) - \|\hat{\mathbf{P}}_{\gamma_0^m}^- v\|_m^2 + \hat{\mathcal{L}}_{\Delta\gamma_0^m}^\circ(\hat{\mathbf{P}}_{\gamma_0^m}^0 v) \\ &\equiv \hat{\beta}_{\gamma_0^m}(v) + \hat{\alpha}_{\gamma_0^m}(v) \quad \forall v \in X\hat{N}\mathcal{O}(\sqrt{m}\epsilon)_{\gamma_0^m}\end{aligned}$$

as in Theorem 5.2, where $\hat{\alpha}_{\gamma_0^m}(v) = \hat{\mathcal{L}}_{\Delta\gamma_0^m}^{\circ}(\hat{\mathbf{P}}_{\gamma_0^m}^0 v)$.

Recall that the C^1 -maps $\mathfrak{h}_{\gamma_0} : \mathbf{B}_{\epsilon}(\mathbf{H}^0(B_{\gamma_0})) \rightarrow \mathbf{H}^-(B_{\gamma_0}) + \mathbf{H}^+(B_{\gamma_0}) \cap XN\mathcal{O}_{\gamma_0}$ and $\hat{\mathfrak{h}}_{\gamma_0^m} : \mathbf{B}_{\sqrt{m}\epsilon}(\mathbf{H}^0(\hat{B}_{\gamma_0^m})) \rightarrow \mathbf{H}^-(\hat{B}_{\gamma_0^m}) + \mathbf{H}^+(\hat{B}_{\gamma_0^m}) \cap X\hat{N}\varphi_m(\mathcal{O})_{\gamma_0^m}$ are uniquely determined by the equations

$$(\mathbf{P}_{\gamma_0}^- + \mathbf{P}_{\gamma_0}^+)A_{\gamma_0}(\xi + \mathfrak{h}_{\gamma_0}(\xi)) = 0 \quad \forall \xi \in \mathbf{H}^0(B)(\epsilon)_{\gamma_0}, \quad (6.15)$$

$$(\hat{\mathbf{P}}_{\gamma_0^m}^- + \hat{\mathbf{P}}_{\gamma_0^m}^+)\hat{A}_{\gamma_0^m}(\xi + \hat{\mathfrak{h}}_{\gamma_0^m}(\xi)) = 0 \quad \forall \xi \in \mathbf{H}^0(\hat{B})(\sqrt{m}\epsilon)_{\gamma_0^m}, \quad (6.16)$$

$$\mathfrak{h}_{\gamma_0}(0) = 0, \quad \hat{\mathfrak{h}}_{\gamma_0^m}(0) = 0$$

with the implicit function theorem. By (6.14) we have

$$\tilde{\varphi}_m(\mathbf{H}^{\star}(B_{\gamma_0})) \subset \mathbf{H}^{\star}(\hat{B}_{\gamma_0^m}), \quad \star = -, 0, +. \quad (6.17)$$

Now (6.15) implies that $A_{\gamma_0}(\xi + \mathfrak{h}_{\gamma_0}(\xi)) \in \mathbf{H}^0(B_{\gamma_0})$ for any $\xi \in \mathbf{H}^0(B)(\epsilon)_{\gamma_0}$. This and (6.13) lead to

$$\mathbf{H}^0(\hat{B}_{\gamma_0^m}) \ni \tilde{\varphi}_m(A_{\gamma_0}(\xi + \mathfrak{h}_{\gamma_0}(\xi))) = \hat{A}_{\gamma_0^m}(\tilde{\varphi}_m(\xi) + \tilde{\varphi}_m(\mathfrak{h}_{\gamma_0}(\xi)))$$

and hence

$$(\hat{\mathbf{P}}_{\gamma_0^m}^- + \hat{\mathbf{P}}_{\gamma_0^m}^+)\hat{A}_{\gamma_0^m}(\tilde{\varphi}_m(\xi) + \tilde{\varphi}_m(\mathfrak{h}_{\gamma_0}(\xi))) = 0 \quad \forall \xi \in \mathbf{H}^0(B)(\epsilon)_{\gamma_0}.$$

By the implicit function theorem this and (6.16) imply that

$$\hat{\mathfrak{h}}_{\gamma_0^m}(\tilde{\varphi}_m(v)) = \tilde{\varphi}_m(\mathfrak{h}_{\gamma_0}(v)) \quad \forall v \in \mathbf{H}^0(B)(\epsilon)_{\gamma_0}. \quad (6.18)$$

It follows from this and (6.11) that

$$\begin{aligned} \hat{\mathcal{L}}_{\Delta\gamma_0^m}^{\circ}(\xi^m) &= \hat{\mathcal{F}}_{\gamma_0^m}(\xi^m + \hat{\mathfrak{h}}_{\gamma_0^m}(\xi^m)) \\ &= \hat{\mathcal{F}}_{\gamma_0^m}(\xi^m + \tilde{\varphi}_m(\mathfrak{h}_{\gamma_0}(\xi))) \\ &= m^2 \mathcal{F}_{\gamma_0}(\xi + \mathfrak{h}_{\gamma_0}(\xi)) \\ &= m^2 \mathcal{L}_{\Delta\gamma_0}^{\circ}(\xi) \quad \forall \xi \in \mathbf{H}^0(B)(\epsilon)_{\gamma_0}. \end{aligned} \quad (6.19)$$

This, (6.17) and (6.14) lead to

$$\hat{\alpha}_{\gamma_0^m}(\tilde{\varphi}_m(v)) = m^2 \alpha_{\gamma_0}(v) \quad \forall v \in \mathbf{P}_{\gamma_0}^0(XN\mathcal{O}(\epsilon)_{\gamma_0}), \quad (6.20)$$

$$\hat{\beta}_{\gamma_0^m}(\tilde{\varphi}_m(v)) = m^2 \beta_{\gamma_0}(v) \quad \forall v \in (\mathbf{P}_{\gamma_0}^+ + \mathbf{P}_{\gamma_0}^-)(XN\mathcal{O}(\epsilon)_{\gamma_0}). \quad (6.21)$$

Carefully checking the proof of Theorem 2.5 in [26] it follows from (6.11)-(6.14) that

$$\hat{\Psi}_{\gamma_0^m}(\tilde{\varphi}_m(v)) = \tilde{\varphi}_m(\Psi_{\gamma_0}(v)) \quad \forall v \in XN\mathcal{O}(\epsilon)_{\gamma_0}. \quad (6.22)$$

Take $0 < \epsilon_1 < \epsilon_2 \ll \epsilon$ such that

$$\begin{aligned} R_{\gamma_0} &:= \mathbf{H}^0(B)(\epsilon_1)_{\gamma_0} \oplus \mathbf{H}^-(B)(\epsilon_1)_{\gamma_0} \oplus (\mathbf{H}^+(B)_{\gamma_0} \cap XN\mathcal{O}(\epsilon_1)_{\gamma_0}) \subset XN\mathcal{O}(\epsilon)_{\gamma_0}, \\ \tilde{\varphi}_m(R_{\gamma_0}) &\subset X\hat{N}\mathcal{O}(\sqrt{m}\epsilon)_{\gamma_0^m}, \\ \tilde{\varphi}_m(\mathbf{H}^+(B)_{\gamma_0} \cap XN\mathcal{O}(\epsilon_1)_{\gamma_0}) &\subset \mathbf{H}^+(\hat{B})_{\gamma_0^m} \cap X\hat{N}\tilde{\varphi}_m(\mathcal{O})(\epsilon_2)_{\gamma_0^m}, \\ \hat{R}_{\gamma_0^m} &:= \tilde{\varphi}_m(\mathbf{H}^0(B)(\epsilon_1)_{\gamma_0}) \hat{\oplus} \tilde{\varphi}_m(\mathbf{H}^-(B)(\epsilon_1)_{\gamma_0}) \hat{\oplus} (\mathbf{H}^+(\hat{B})_{\gamma_0^m} \cap X\hat{N}\tilde{\varphi}_m(\mathcal{O})(\epsilon_2)_{\gamma_0^m}) \\ &\subset X\hat{N}\tilde{\varphi}_m(\mathcal{O})(\sqrt{m}\epsilon)_{\gamma_0^m}. \end{aligned}$$

Set $R = S^1 \cdot R_{\gamma_0}$ and $\hat{R} = S^1 \cdot \hat{R}_{\gamma_0^m}$. By (6.22) we have the commutative diagrams

$$\begin{array}{ccc} R & \xrightarrow{\Psi} & \Psi(R) \subset XN\mathcal{O}(\epsilon) \\ \tilde{\varphi}_m \downarrow & & \downarrow \tilde{\varphi}_m \\ \hat{R} & \xrightarrow{\hat{\Psi}} & \hat{\Psi}(\hat{R}) \subset X\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon). \end{array}$$

and thus

$$\begin{array}{ccc} H_*((\mathcal{F}^X \circ \Psi)_c \cap R, (\mathcal{F}^X \circ \Psi)_c \cap (R \setminus \mathcal{O}); \mathbb{K}) & \xrightarrow{\Psi_*} & H_*((\mathcal{F}^X)_c \cap \Psi(R), (\mathcal{F}^X)_c \cap (\Psi(R) \setminus \mathcal{O}); \mathbb{K}) \\ (\tilde{\varphi}_m)_* \downarrow & & \downarrow (\tilde{\varphi}_m)_* \\ H_*((\hat{\mathcal{F}}^X \circ \hat{\Psi})_{m^2c} \cap \hat{R}, (\hat{\mathcal{F}}^X \circ \hat{\Psi})_{m^2c} \cap (\hat{R} \setminus \varphi_m(\mathcal{O})); \mathbb{K}) & \xrightarrow{(\hat{\Psi})_*} & H_*((\hat{\mathcal{F}}^X)_{m^2c} \cap \hat{\Psi}(\hat{R}), (\hat{\mathcal{F}}^X)_{m^2c} \cap (\hat{\Psi}(\hat{R}) \setminus \varphi_m(\mathcal{O})); \mathbb{K}). \end{array}$$

Now $\mathcal{F}^X \circ \Psi = \beta + \alpha$, $\hat{\mathcal{F}}^X \circ \hat{\Psi} = \hat{\beta} + \hat{\alpha}$, and Ψ_* , $\hat{\Psi}_*$ are isomorphisms.² By the assumptions that $m^-(\mathcal{O}) = m^-(\varphi_m(\mathcal{O}))$ and $m^0(\mathcal{O}) = m^0(\varphi_m(\mathcal{O}))$, \hat{R} is an open neighborhood of $\varphi_m(\mathcal{O})$ in $X\hat{N}\varphi_m(\mathcal{O})(\sqrt{m}\epsilon)$. Take $\hat{V}_1 = \Psi(R)$ and $\hat{V}_m = \hat{\Psi}(\hat{R})$ in Proposition 6.6. We get that Proposition 6.6 is equivalent to

Claim 6.7 $\tilde{\varphi}_m$ induces isomorphisms

$$H_*((\beta + \alpha)_c \cap R, (\beta + \alpha)_c \cap (R \setminus \mathcal{O}); \mathbb{K}) \rightarrow H_*((\hat{\beta} + \hat{\alpha})_{m^2c} \cap \hat{R}, (\hat{\beta} + \hat{\alpha})_{m^2c} \cap (\hat{R} \setminus \varphi_m(\mathcal{O})); \mathbb{K}).$$

Since the deformation retracts

$$\begin{aligned} \mathbf{H}^0(B) \oplus \mathbf{H}^-(B) \oplus (\mathbf{H}^+(B) \cap XN\mathcal{O}) \times [0, 1] &\rightarrow \mathbf{H}^0(B) \oplus \mathbf{H}^-(B) \oplus (\mathbf{H}^+(B) \cap XN\mathcal{O}) \\ (x, v^0 + v^- + v^+) &\mapsto (x, v^0 + v^- + tv^+), \\ \mathbf{H}^0(\hat{B}) \hat{\oplus} \mathbf{H}^-(\hat{B}) \hat{\oplus} (\mathbf{H}^+(\hat{B}) \cap X\hat{N}\varphi_m(\mathcal{O})) \times [0, 1] &\rightarrow \mathbf{H}^0(\hat{B}) \hat{\oplus} \mathbf{H}^-(\hat{B}) \hat{\oplus} (\mathbf{H}^+(\hat{B}) \cap X\hat{N}\varphi_m(\mathcal{O})) \\ (x, v^0 + v^- + v^+) &\mapsto (x, v^0 + v^- + tv^+) \end{aligned}$$

commute with $\tilde{\varphi}_m$, Claim 6.7 is equivalent to

Claim 6.8 $\tilde{\varphi}_m$ induces isomorphisms from

$$\begin{aligned} H_*((\beta + \alpha)_c \cap \square, (\beta + \alpha)_c \cap (\square \setminus \mathcal{O}); \mathbb{K}) &\text{ to} \\ H_*((\hat{\beta} + \hat{\alpha})_{m^2c} \cap \tilde{\varphi}_m(\square), (\hat{\beta} + \hat{\alpha})_{m^2c} \cap ((\tilde{\varphi}_m(\square) \setminus \varphi_m(\mathcal{O})); \mathbb{K}). \end{aligned}$$

Here $\square = \mathbf{H}^0(B)(\epsilon_1) \oplus \mathbf{H}^-(B)(\epsilon_1)$.

Since $\tilde{\varphi}_m : \square \rightarrow \tilde{\varphi}_m(\square)$ is a linear diffeomorphism and

$$(\hat{\beta} + \hat{\alpha})(\tilde{\varphi}_m(\xi)) = m^2(\beta + \alpha)(\xi) \quad \forall \xi \in \square$$

because of (6.20) and (6.21). Claim 6.8 follows immediately. Theorem 6.1 is proved.

□

²It is where the assumptions are used.

6.2 Proof of Theorem 6.3

Step 1. Proving the first equality in (6.1). Since $m^0(\mathcal{O}) = m^0(\varphi_m(\mathcal{O}))$, $\tilde{\varphi}_m$ restricts to a linear homeomorphism

$$\tilde{\varphi}_m^x : \mathbf{H}^0(B)_x \rightarrow \mathbf{H}^0(\hat{B})_{x^m}$$

for any $x \in \mathcal{O}$. By (6.19) it also restricts to homeomorphism

$$(\mathcal{L}_{\Delta x}^\circ)_c \cap \mathbf{H}^0(B)(\epsilon/\sqrt{m})_x \rightarrow (\hat{\mathcal{L}}_{\Delta x^m}^\circ)_{m^2 c} \cap \mathbf{H}^0(\hat{B})(\epsilon)_{x^m}, \quad (6.23)$$

which induces isomorphisms

$$\begin{aligned} & H_*((\mathcal{L}_{\Delta x}^\circ)_c \cap \mathbf{H}^0(B)(\epsilon/\sqrt{m})_x, (\mathcal{L}_{\Delta x}^\circ)_c \cap \mathbf{H}^0(B)(\epsilon/\sqrt{m})_x \setminus \{0\}; \mathbb{K}) \\ & \rightarrow H_*((\hat{\mathcal{L}}_{\Delta x^m}^\circ)_{m^2 c} \cap \mathbf{H}^0(\hat{B})(\epsilon)_{x^m}, (\hat{\mathcal{L}}_{\Delta x^m}^\circ)_{m^2 c} \cap \mathbf{H}^0(\hat{B})(\epsilon)_{x^m} \setminus \{0\}; \mathbb{K}). \end{aligned}$$

In particular the first equality in (6.1) holds true.

Step 2. Proving (6.2). Write $x = \gamma_0$. Recall that $S^1 = \mathbb{R}/\mathbb{Z} = \{[s] \mid [s] = s + \mathbb{Z}, s \in \mathbb{R}\}$. Since S_x^1 is a finite cyclic subgroup of S^1 , we may assume $S_x^1 = \{[j/l] \mid j = 0, \dots, l-1\}$ for some $l \in \mathbb{N}$. Then $S_{x^m}^1 = \{[\frac{j}{ml}] \mid j = 0, \dots, ml-1\}$. Clearly, $\tilde{\varphi}_m^x([1/l] \cdot \xi) = [\frac{1}{ml}] \cdot \tilde{\varphi}_m^x(\xi)$. Obverse that \mathfrak{h}_x (resp. \mathfrak{h}_{x^m}) is S_x^1 (resp. $S_{x^m}^1$) equivariant. From (6.18) and (6.19) we derive that the homeomorphism in (6.23) induces a homeomorphism

$$(\mathcal{L}_{\Delta x}^\circ)_c \cap \mathbf{H}^0(B)(\epsilon/\sqrt{m})_x / S_x^1 \rightarrow (\hat{\mathcal{L}}_{\Delta x^m}^\circ)_{m^2 c} \cap \mathbf{H}^0(\hat{B})(\epsilon)_{x^m} / S_{x^m}^1$$

and therefore isomorphisms

$$\begin{aligned} & H_*((\mathcal{L}_{\Delta x}^\circ)_c \cap \mathbf{H}^0(B)(\epsilon/\sqrt{m})_x / S_x^1, ((\mathcal{L}_{\Delta x}^\circ)_c \cap \mathbf{H}^0(B)(\epsilon/\sqrt{m})_x \setminus \{0\}) / S_x^1; \mathbb{K}) \\ & \rightarrow H_*((\hat{\mathcal{L}}_{\Delta x^m}^\circ)_{m^2 c} \cap \mathbf{H}^0(\hat{B})(\epsilon)_{x^m} / S_{x^m}^1, ((\hat{\mathcal{L}}_{\Delta x^m}^\circ)_{m^2 c} \cap \mathbf{H}^0(\hat{B})(\epsilon)_{x^m} \setminus \{0\}) / S_{x^m}^1; \mathbb{K}). \end{aligned}$$

As in the arguments below (5.24), this may induce isomorphisms

$$\begin{aligned} & H_*((\mathcal{L}_{\Delta x}^\circ)_c \cap \mathbf{H}^0(B)(\epsilon/\sqrt{m})_x, (\mathcal{L}_{\Delta x}^\circ)_c \cap \mathbf{H}^0(B)(\epsilon/\sqrt{m})_x \setminus \{0\}; \mathbb{K})^{S_x^1} \\ & \rightarrow H_*((\hat{\mathcal{L}}_{\Delta x^m}^\circ)_{m^2 c} \cap \mathbf{H}^0(\hat{B})(\epsilon)_{x^m}, (\hat{\mathcal{L}}_{\Delta x^m}^\circ)_{m^2 c} \cap \mathbf{H}^0(\hat{B})(\epsilon)_{x^m} \setminus \{0\}; \mathbb{K})^{S_{x^m}^1} \end{aligned}$$

if the characteristic of \mathbb{K} is zero or prime to orders of $S_{\gamma_0}^1$ and $S_{\gamma_0^m}^1$, and hence (6.2).

Step 3. Proving the second equality in (6.1). Let $m^- = m^-(\mathcal{O}) = m^-(\varphi_m(\mathcal{O}))$. Recall that $\hat{\mathcal{F}}_{\gamma_0^m}^X$ is the restriction of $\hat{\mathcal{F}}_{\gamma_0^m}$ to $\hat{N}(\tilde{\varphi}_m(\mathcal{O}))(\sqrt{m}\delta)_{\gamma_0^m} \cap T_{\gamma_0^m} \mathcal{X}$. According to the illustrations above Theorem 1.10 we define

$${}^m\mathcal{F}_{\gamma_0^m} = \mathcal{L} \circ \text{EXP}|_{N(\tilde{\varphi}_m(\mathcal{O}))(\delta)_{\gamma_0^m}}, \quad {}^m\mathcal{F}_{\gamma_0^m}^X = \mathcal{L} \circ \text{EXP}|_{N(\tilde{\varphi}_m(\mathcal{O}))(\delta)_{\gamma_0^m} \cap T_{\gamma_0^m} \mathcal{X}}$$

and ${}^m\mathcal{L}_{\Delta \gamma_0^m}^\circ : \mathbf{H}^0({}^mB)(\epsilon)_{\gamma_0^m} \rightarrow \mathbb{R}$ by

$${}^m\mathcal{L}_{\Delta \gamma_0^m}^\circ(v) = \mathcal{L} \circ \text{EXP}_{\gamma_0^m}(v + {}^m\mathfrak{h}_{\gamma_0^m}(v)).$$

Then the functionals ${}^m\mathcal{F}_{\gamma_0^m}$ and ${}^m\mathcal{F}_{\gamma_0^m}^X$ satisfy Theorem A.1 and Theorem A.5 around $0 = 0_{\gamma_0^m}$, respectively. Applying Corollary A.6 to ${}^m\mathcal{F}_{\gamma_0^m}^X$ and $\hat{\mathcal{F}}_{\gamma_0^m}^X$ gives

$$\begin{aligned} C_{q-m-}({}^m\mathcal{L}_{\Delta\gamma_0^m}^\circ, 0; \mathbb{K}) &= C_q({}^m\mathcal{F}_{\gamma_0^m}^X, 0; \mathbb{K}) \quad \text{and} \\ C_{q-m-}(\hat{\mathcal{L}}_{\Delta\gamma_0^m}^\circ, 0; \mathbb{K}) &= C_q(\hat{\mathcal{F}}_{\gamma_0^m}^X, 0; \mathbb{K}) \end{aligned}$$

for any $q \in \mathbb{N} \cup \{0\}$. So it suffices to prove that

$$C_*({}^m\mathcal{F}_{\gamma_0^m}^X, 0; \mathbb{K}) = C_*(\hat{\mathcal{F}}_{\gamma_0^m}^X, 0; \mathbb{K}). \quad (6.24)$$

Since $C_*({}^m\mathcal{F}_{\gamma_0^m}^X, 0; \mathbb{K}) = C_*(\mathcal{L}|_{\mathbf{S}}, \gamma_0^m; \mathbb{K})$ and $C_*(\hat{\mathcal{F}}_{\gamma_0^m}^X, 0; \mathbb{K}) = C_*(\mathcal{L}|_{\hat{\mathbf{S}}}, \gamma_0^m; \mathbb{K})$, where

$$\begin{aligned} \mathbf{S} &= \text{EXP}|_{N(\tilde{\varphi}_m(\mathcal{O}))(\delta)_{\gamma_0^m}} (N(\tilde{\varphi}_m(\mathcal{O}))(\sqrt{m}\delta)_{\gamma_0^m} \cap T_{\gamma_0^m}\mathcal{X}), \\ \hat{\mathbf{S}} &= \text{EXP}|_{\hat{N}(\tilde{\varphi}_m(\mathcal{O}))(\sqrt{m}\delta)_{\gamma_0^m}} (\hat{N}(\tilde{\varphi}_m(\mathcal{O}))(\sqrt{m}\delta)_{\gamma_0^m} \cap T_{\gamma_0^m}\mathcal{X}) \end{aligned}$$

are two at least C^{k-1} submanifolds of codimension one in \mathcal{X} and both are transversely intersecting at γ_0^m with $S^1 \cdot \gamma_0^m = \varphi_m(\mathcal{O})$, we only need to prove that

$$C_*(\mathcal{L}|_{\mathbf{S}}, \gamma_0^m; \mathbb{K}) = C_*(\mathcal{L}|_{\hat{\mathbf{S}}}, \gamma_0^m; \mathbb{K}). \quad (6.25)$$

Note that the S^1 -action on \mathcal{X} is C^1 . We get a C^1 map

$$\mathfrak{T} : S^1 \times \mathbf{S} \rightarrow \mathcal{X}, \quad ([s], x) \mapsto [s] \cdot x.$$

Since \mathcal{S} is transversal to $\varphi_m(\mathcal{O})$ at γ_0^m , the differential $d\mathfrak{T}([0], \gamma_0^m)$ is an isomorphism. So there exists a neighborhood $\mathcal{N}(\gamma_0^m)$ of γ_0^m in \mathbf{S} and $0 < r \ll 1/2$ such that \mathfrak{T} is a diffeomorphism from $\{[s] \mid s \in (-r, r)\} \times \mathcal{N}(\gamma_0^m)$ onto a neighborhood of γ_0^m in \mathcal{X} . Because $\hat{\mathbf{S}}$ is transversal to $\varphi_m(\mathcal{O})$ at γ_0^m as well $\mathfrak{T}^{-1}(\hat{\mathbf{S}})$ is a submanifold in $\{[s] \mid s \in (-r, r)\} \times \mathcal{N}(\gamma_0^m)$ which is transversal to $\{[s] \mid s \in (-r, r)\} \times \{\gamma_0^m\}$. It follows that there exists a C^1 -map $\mathfrak{S} : \mathcal{N}(\gamma_0^m) \rightarrow \{[s] \mid s \in (-r, r)\}$ such that the graph $\text{Gr}(\mathfrak{S})$ is a neighborhood of $\gamma_0^m \equiv ([0], \gamma_0^m)$ in $\mathfrak{T}^{-1}(\hat{\mathbf{S}})$ by shrinking $\{[s] \mid s \in (-r, r)\} \times \mathcal{N}(\gamma_0^m)$ if necessary. This implies that the composition

$$\Theta : \mathcal{N}(\gamma_0^m) \ni x \mapsto (\mathfrak{S}(x), x) \in \text{Gr}(\mathfrak{S}) \xrightarrow{\mathfrak{T}} \mathcal{X}$$

is a C^1 diffeomorphism from $\mathcal{N}(\gamma_0^m)$ onto a neighborhood $\mathcal{W}(\gamma_0^m)$ of γ_0^m in $\hat{\mathbf{S}}$. Clearly, $\Theta(\gamma_0^m) = \gamma_0^m$ and $\mathcal{L}|_{\hat{\mathbf{S}}}(\Theta(x)) = \mathcal{L}([\mathfrak{S}(x)] \cdot x) = \mathcal{L}(x) = \mathcal{L}|_{\mathbf{S}}(x)$ for any $x \in \mathcal{N}(\gamma_0^m)$. Hence Θ induces an isomorphism from $C_*(\mathcal{L}|_{\mathbf{S}}, \gamma_0^m; \mathbb{K})$ to $C_*(\mathcal{L}|_{\hat{\mathbf{S}}}, \gamma_0^m; \mathbb{K})$. This proves (6.25), and hence (6.24) and the second equality in (6.1).

Step 4. Proving (6.3). Recall that $S_{\gamma_0^m}^1 = \{[\frac{j}{ml}] \mid j = 0, \dots, ml-1\}$ for some $l \in \mathbb{N}$. It is easily checked that $S_{\gamma_0^m}^1$ acts on

$$N(\tilde{\varphi}_m(\mathcal{O}))(\delta)_{\gamma_0^m} \cap T_{\gamma_0^m}\mathcal{X} \quad (\text{resp. } \hat{N}(\tilde{\varphi}_m(\mathcal{O}))(\sqrt{m}\delta)_{\gamma_0^m} \cap T_{\gamma_0^m}\mathcal{X})$$

and therefore on \mathbf{S} (resp. $\hat{\mathbf{S}}$). We shrink the above $r > 0$ so that $r < \frac{1}{4ml}$. Take a $S_{\gamma_0^m}^1$ -invariant open neighborhood $\mathcal{N}(\gamma_0^m)^*$ of γ_0^m in \mathbf{S} such that $\mathcal{N}(\gamma_0^m)^* \subset \mathcal{N}(\gamma_0^m)$.

Given a $x \in \mathcal{N}(\gamma_0^m)^*$, by the construction of Θ we have $\Theta(x) = [\mathfrak{S}(x)] \cdot x$ and

$$\begin{aligned}\Theta([\tfrac{1}{ml}] \cdot x) &= [\mathfrak{S}([\tfrac{1}{ml}] \cdot x)] \cdot ([\tfrac{1}{ml}] \cdot x) \\ &= [\mathfrak{S}([\tfrac{1}{ml}] \cdot x) + \tfrac{1}{ml}] \cdot x \\ &= [\mathfrak{S}([\tfrac{1}{ml}] \cdot x) - \mathfrak{S}(x) + \tfrac{1}{ml}] \cdot \Theta(x).\end{aligned}$$

Note that $\mathfrak{S}([\tfrac{1}{ml}] \cdot x)$ and $-\mathfrak{S}(x)$ belong to $(-r, r)$, and that $0 < r < \frac{1}{4ml}$. We deduce that $\mathfrak{S}([\tfrac{1}{ml}] \cdot x) - \mathfrak{S}(x) + \frac{1}{ml}$ sits in $(-\frac{1}{ml}, \frac{2}{ml})$. Moreover, it is also one of the numbers $\frac{j}{ml}$, $j = 0, 1, \dots, ml - 1$. Hence $\mathfrak{S}([\tfrac{1}{ml}] \cdot x) - \mathfrak{S}(x) + \frac{1}{ml} = \frac{1}{ml}$ or 0. If $x \neq \gamma_0^m$ the second case cannot occur because Θ is a diffeomorphism. This shows that $\Theta([\tfrac{1}{ml}] \cdot x) = [\tfrac{1}{ml}] \cdot \Theta(x)$ for any $x \in \mathcal{N}(\gamma_0^m)^*$. Namely, Θ is an equivariant diffeomorphism from $\mathcal{N}(\gamma_0^m)^*$ to some $S_{\gamma_0^m}^1$ -invariant open neighborhood of γ_0^m in $\hat{\mathbf{S}}$. It follows that there exist $S_{\gamma_0^m}^1$ -invariant open neighborhoods of γ_0^m in \mathbf{S} and $\hat{\mathbf{S}}$, U and \hat{U} , such that

$$\begin{aligned}& H_* \left((\mathcal{L}|_{\mathbf{S}})_c \cap U / S_{\gamma_0^m}^1; (\mathcal{L}|_{\hat{\mathbf{S}}})_c \cap (U \setminus \{\gamma_0^m\}) / S_{\gamma_0^m}^1; \mathbb{K} \right) \\ &= H_* \left((\mathcal{L}|_{\hat{\mathbf{S}}})_c \cap \hat{U} / S_{\gamma_0^m}^1; (\mathcal{L}|_{\mathbf{S}})_c \cap (\hat{U} \setminus \{\gamma_0^m\}) / S_{\gamma_0^m}^1; \mathbb{K} \right).\end{aligned}$$

This implies that there exist $S_{\gamma_0^m}^1$ -invariant open neighborhoods V and \hat{V} of 0 in $N(\tilde{\varphi}_m(\mathcal{O}))(\delta)_{\gamma_0^m} \cap T_{\gamma_0^m} \mathcal{X}$ and $\hat{N}(\tilde{\varphi}_m(\mathcal{O}))(\sqrt{m}\delta)_{\gamma_0^m} \cap T_{\gamma_0^m} \mathcal{X}$ respectively, such that

$$\begin{aligned}& H_* \left(({}^m\mathcal{F}_{\gamma_0^m}^X)_c \cap V / S_{\gamma_0^m}^1; ({}^m\mathcal{F}_{\gamma_0^m}^X)_c \cap (V \setminus \{\gamma_0^m\}) / S_{\gamma_0^m}^1; \mathbb{K} \right) \\ &= H_* \left((\hat{\mathcal{F}}_{\gamma_0^m}^X)_c \cap \hat{V} / S_{\gamma_0^m}^1; (\hat{\mathcal{F}}_{\gamma_0^m}^X)_c \cap (\hat{V} \setminus \{\gamma_0^m\}) / S_{\gamma_0^m}^1; \mathbb{K} \right).\end{aligned}\quad (6.26)$$

For the field \mathbb{K} of characteristic zero or prime to order of $S_{\gamma_0^m}^1$, as in the proof of the equality above (5.25) using Theorem A.5 (for ${}^m\mathcal{F}_{\gamma_0^m}^X$ and $\hat{\mathcal{F}}_{\gamma_0^m}^X$) we can deduce that

$$H_q \left(({}^m\mathcal{F}_{\gamma_0^m}^X)_c \cap V / S_{\gamma_0^m}^1; ({}^m\mathcal{F}_{\gamma_0^m}^X)_c \cap (V \setminus \{\gamma_0^m\}) / S_{\gamma_0^m}^1; \mathbb{K} \right) \quad (6.27)$$

$$\begin{aligned}&= \left(H_{m-}(\mathbf{H}^-({}^mB)_{\gamma_0^m}, \mathbf{H}^-({}^mB)_{\gamma_0^m} \setminus \{0\}; \mathbb{K}) \otimes C_{q-m-}(\mathcal{L}_{\Delta\gamma_0^m}^\circ, 0; \mathbb{K}) \right)^{S_{\gamma_0^m}^1}, \\ & H_* \left((\hat{\mathcal{F}}_{\gamma_0^m}^X)_c \cap \hat{V} / S_{\gamma_0^m}^1; (\hat{\mathcal{F}}_{\gamma_0^m}^X)_c \cap (\hat{V} \setminus \{\gamma_0^m\}) / S_{\gamma_0^m}^1; \mathbb{K} \right) \quad (6.28) \\ &= \left(H_{m-}(\mathbf{H}^-(\hat{B})_{\gamma_0^m}, \mathbf{H}^-(\hat{B})_{\gamma_0^m} \setminus \{0\}; \mathbb{K}) \otimes C_{q-m-}(\hat{\mathcal{L}}_{\Delta\gamma_0^m}^\circ, 0; \mathbb{K}) \right)^{S_{\gamma_0^m}^1}\end{aligned}$$

for each $q = 0, 1, \dots$. Obverse that $\mathbf{H}^-({}^mB)_{\gamma_0^m}$ and $\mathbf{H}^-(\hat{B})_{\gamma_0^m}$ are orthogonal complementary of $\dot{\gamma}_0^m \mathbb{R}$ in $\mathbf{H}^-(d^2\mathcal{L}|_{\mathcal{X}}(\gamma_0^m)) \subset T_{\gamma_0^m} \mathcal{X}$ with respect to the inner product $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_m$ in (5.3), respectively. Moreover, we have an equivalent inner product homotopy from $\langle \cdot, \cdot \rangle_1$ to $\langle \cdot, \cdot \rangle_m$ given by

$$\langle \xi, \eta \rangle_{(1-\tau)+\tau m} = (1-\tau + \tau m^2) \int_0^1 \langle \xi(t), \eta(t) \rangle dt + \int_0^1 \langle \nabla_\gamma^h \xi(t), \nabla_\gamma^h \eta(t) \rangle dt,$$

where $\tau \in [0, 1]$. Hence a generator of the $S_{\gamma_0^m}^1$ -action on $\mathbf{H}^-(^mB)_{\gamma_0^m}$ reverses orientation if and only if its action on $\mathbf{H}^-(\hat{B})_{\gamma_0^m}$ reverses orientation. This and (6.26)-(6.28) imply

$$C_*(^m\mathcal{L}_{\Delta\gamma_0^m}^\circ, 0; \mathbb{K})^{/S_{\gamma_0^m}^1} = C_*(\hat{\mathcal{L}}_{\Delta\gamma_0^m}^\circ, 0; \mathbb{K})^{/S_{\gamma_0^m}^1}.$$

(6.3) follows from this and (6.2). Theorem 6.3 is proved. \square

7 Computations of S^1 -critical groups

Let Γ be a subgroup of S^1 and let $A \subset \Lambda M$ be a Γ -invariant subset. We denote by A/Γ the quotient space of A with respect to the action of Γ . Rademacher [42, §6.1] defined S^1 -critical group of γ_0 by

$$\overline{C}_*(\mathcal{L}, \gamma_0; \mathbb{K}) = H_*((\Lambda(\gamma_0) \cup S^1 \cdot \gamma_0)/S^1, \Lambda(\gamma_0)/S^1; \mathbb{K}),$$

which is important in studies of closed geodesics. It was proved in [2, (3.10)] that the groups $\overline{C}_*(\mathcal{L}, \gamma_0; \mathbb{K})$ and $C_*(\mathcal{L}, \gamma_0; \mathbb{K}) := H_*(\Lambda(\gamma_0) \cup \{\gamma_0\}, \Lambda(\gamma_0); \mathbb{K})$ have relations

$$\overline{C}_*(\mathcal{L}, \gamma_0; \mathbb{K}) = C_*(\mathcal{L}, \gamma_0; \mathbb{K})^{S_{\gamma_0}^1}. \quad (7.1)$$

Let us outline a proof of it with our previous methods. For a small $\epsilon > 0$ let

$$H_{2\epsilon} = \{\xi \in \mathbf{B}_{2\epsilon}(T_{\gamma_0}\Lambda M) \mid \langle \dot{\gamma}_0, \xi \rangle_0 = 0\}.$$

By Lemma 2.2.8 of [27] the set $\Sigma_{2\epsilon} = \text{EXP}(H_{2\epsilon}) = \{\exp_{\gamma_0}(\xi) \mid \xi \in H_{2\epsilon}\}$ is a slice of the S^1 -action on ΛM . This means: (i) $[s] \cdot \Sigma_{2\epsilon} = \Sigma_{2\epsilon} \forall [s] \in S_{\gamma_0}^1$, (ii) if $([s] \cdot \Sigma_{2\epsilon}) \cap \Sigma_{2\epsilon} \neq \emptyset$ then $[s] \in S_{\gamma_0}^1$, (iii) $\exists 0 < \nu \ll 1$ such that $(-\nu, \nu) \times \Sigma_{2\epsilon} \ni (s, x) \mapsto [s] \cdot x \in \Lambda M$ is a homeomorphism onto an open neighborhood of γ_0 in ΛM . The latter implies that $\Sigma_{2\epsilon}/S_{\gamma_0}^1 \ni S_{\gamma_0}^1 \cdot x \rightarrow S^1 \cdot x \in \Lambda M/S^1$ is a homeomorphism onto $(S^1 \cdot \Sigma_{2\epsilon})/S^1$ which is an open neighborhood of the equivalence class of γ_0 in $\Lambda M/S^1$. It follows that

$$\begin{aligned} \overline{C}_*(\mathcal{L}, \gamma_0; \mathbb{K}) &= H_*((\Lambda(\gamma_0) \cap \Sigma_{2\epsilon} \cup \{\gamma_0\})/S_{\gamma_0}^1, \Lambda(\gamma_0) \cap \Sigma_{2\epsilon}/S_{\gamma_0}^1; \mathbb{K}) \\ &= H_*(\mathcal{L}_c \cap \overline{\Sigma_{2\epsilon}}/S_{\gamma_0}^1, \mathcal{L}_c \cap \overline{\Sigma_{2\epsilon}} \setminus \{\gamma_0\}/S_{\gamma_0}^1; \mathbb{K}), \end{aligned} \quad (7.2)$$

where $\overline{\Sigma_{2\epsilon}} = \text{EXP}(\overline{H_{2\epsilon}})$ is the closure of $\Sigma_{2\epsilon}$. Since $\overline{\Sigma_{2\epsilon}}/S_{\gamma_0}^1$ is a complete metric space, using the stability of critical groups of isolated critical point for continuous functionals on metric spaces in [21, Th.5.2] or [19, Th.1.5] we may prove that

$$\begin{aligned} &H_*(\mathcal{L}_c \cap \overline{\Sigma_{2\epsilon}}/S_{\gamma_0}^1, \mathcal{L}_c \cap \overline{\Sigma_{2\epsilon}} \setminus \{\gamma_0\}/S_{\gamma_0}^1; \mathbb{K}) \\ &= H_*((\mathcal{L}^*)_c \cap \overline{\Sigma_{2\epsilon}}/S_{\gamma_0}^1, (\mathcal{L}^*)_c \cap \overline{\Sigma_{2\epsilon}} \setminus \{\gamma_0\}/S_{\gamma_0}^1; \mathbb{K}). \end{aligned} \quad (7.3)$$

Let 2ϵ be less than ϵ in (1.12). Using the chart in (1.12) we write $\mathcal{E}^* := \mathcal{L}^* \circ \text{EXP}$. Then $\mathcal{E}^*|_{\overline{H_{2\epsilon}}} = \mathcal{L}^*|_{\overline{\Sigma_{2\epsilon}}} \circ \text{EXP}|_{\overline{H_{2\epsilon}}}$ and

$$\begin{aligned} &H_*((\mathcal{L}^*)_c \cap \overline{\Sigma_{2\epsilon}}/S_{\gamma_0}^1, (\mathcal{L}^*)_c \cap \overline{\Sigma_{2\epsilon}} \setminus \{\gamma_0\}/S_{\gamma_0}^1; \mathbb{K}) \\ &= H_*((\mathcal{E}^*|_{\overline{H_{2\epsilon}}})_c/S_{\gamma_0}^1, (\mathcal{E}^*|_{\overline{H_{2\epsilon}}})_c \setminus \{\gamma_0\}/S_{\gamma_0}^1; \mathbb{K}). \end{aligned} \quad (7.4)$$

Define a family of inner products on $T_{\gamma_0}\Lambda M$ by

$$\langle \xi, \eta \rangle_\tau = \int_0^1 \langle \xi(t), \eta(t) \rangle dt + \tau \int_0^1 \langle \nabla_\gamma^h \xi(t), \nabla_\gamma^h \eta(t) \rangle dt, \quad \tau \in [0, 1].$$

Note that γ_0 is at least C^2 in our case. Denote by $\langle \dot{\gamma}_0 \rangle_\tau^\perp$ the orthogonal complementary of $\dot{\gamma}_0\mathbb{R}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\tau$. Then $\langle \dot{\gamma}_0 \rangle_1^\perp = N\mathcal{O}_{\gamma_0}$ and

$$\langle \dot{\gamma}_0 \rangle_0^\perp = \{ \xi \in T_{\gamma_0}\Lambda M \mid \langle \dot{\gamma}_0, \xi \rangle_0 = 0 \} \quad \text{and hence} \quad H_{2\epsilon} = \mathbf{B}_{2\epsilon}(T_{\gamma_0}\Lambda M) \cap \langle \dot{\gamma}_0 \rangle_0^\perp.$$

For each $\tau \in [0, 1]$ there exists an obvious $S_{\gamma_0}^1$ -equivariant linear diffeomorphism

$$h_\tau : \langle \dot{\gamma}_0 \rangle_0^\perp \rightarrow \langle \dot{\gamma}_0 \rangle_\tau^\perp, \quad \xi \mapsto \xi - \langle \xi, \dot{\gamma}_0 \rangle_\tau \dot{\gamma}_0.$$

And $|h_\tau(\xi)|_1^2 \leq (1 + |\dot{\gamma}_0|_1^2)|\xi|_1^2 \quad \forall \xi \in \langle \dot{\gamma}_0 \rangle_0^\perp$. So we can take $0 < \rho \ll \epsilon$ such that $h_\tau(\xi) \in \mathbf{B}_{2\epsilon}(T_{\gamma_0}\Lambda M)$ for all $\xi \in \mathbf{B}_{2\rho}(T_{\gamma_0}\Lambda M)$. Let $H_{2\epsilon}^\tau = \mathbf{B}_{2\epsilon}(T_{\gamma_0}\Lambda M) \cap \langle \dot{\gamma}_0 \rangle_\tau^\perp$ and consider the pullback $h_\tau^*(\mathcal{E}^*|_{H_{2\epsilon}^\tau})$ for $\tau \in [0, 1]$. They are well-defined on $H_{2\rho} = H_{2\rho}^0$ and have an isolated critical point γ_0 . As above using the stability of critical groups for continuous functionals on metric spaces in [21, Th.5.2] or [19, Th.1.5] we derive

$$\begin{aligned} & H_*((\mathcal{E}^*|_{\overline{H_\epsilon}})_c/S_{\gamma_0}^1, (\mathcal{E}^*|_{\overline{H_\epsilon}})_c \setminus \{\gamma_0\}/S_{\gamma_0}^1; \mathbb{K}) \\ &= H_*((h_0^*\mathcal{E}^*|_{\overline{H_\rho}})_c/S_{\gamma_0}^1, (h_0^*\mathcal{E}^*|_{\overline{H_\rho}})_c \setminus \{\gamma_0\}/S_{\gamma_0}^1; \mathbb{K}) \\ &= H_*((h_1^*\mathcal{E}^*|_{\overline{H_\rho}})_c/S_{\gamma_0}^1, (h_1^*\mathcal{E}^*|_{\overline{H_\rho}})_c \setminus \{\gamma_0\}/S_{\gamma_0}^1; \mathbb{K}) \\ &= H_*((\mathcal{E}^*|_{N\mathcal{O}(\epsilon)\gamma_0})_c/S_{\gamma_0}^1, (\mathcal{E}^*|_{N\mathcal{O}(\epsilon)\gamma_0})_c \setminus \{\gamma_0\}/S_{\gamma_0}^1; \mathbb{K}) \\ &= H_*((\mathcal{F}^*)_c/S_{\gamma_0}^1, (\mathcal{F}^*)_c \setminus \{\gamma_0\}/S_{\gamma_0}^1; \mathbb{K}), \end{aligned} \tag{7.5}$$

where the excision property of relative homology groups are used in the first, second and fourth equality and we may require that ϵ is less than δ in (3.18). By (7.2)-(7.5) we get

$$\begin{aligned} \overline{C}_*(\mathcal{L}, \gamma_0; \mathbb{K}) &= H_*((\Lambda(\gamma_0) \cap \Sigma_{2\epsilon} \cup \{\gamma_0\})/S_{\gamma_0}^1, \Lambda(\gamma_0) \cap \Sigma_{2\epsilon}/S_{\gamma_0}^1; \mathbb{K}) \\ &= H_*((\mathcal{F}^*)_c/S_{\gamma_0}^1, (\mathcal{F}^*)_c \setminus \{\gamma_0\}/S_{\gamma_0}^1; \mathbb{K}). \end{aligned}$$

As in the proofs (5.24) and (5.25) using this and Theorem 1.5(iii) we can derive

Proposition 7.1 *Let \mathbb{K} be a field of characteristic 0 or prime to order $|S_{\gamma_0}^1|$ of $S_{\gamma_0}^1$. Then*

$$\begin{aligned} & \overline{C}_q(\mathcal{L}, \gamma_0; \mathbb{K}) \\ &= \left(H_{m^-(S^1 \cdot \gamma_0)}(\mathbf{H}^-(B)_{\gamma_0}, \mathbf{H}^-(B)_{\gamma_0} \setminus \{0\}; \mathbb{K}) \otimes C_{q-m^-(S^1 \cdot \gamma_0)}(\mathcal{L}_{\Delta\gamma_0}^\circ, 0; \mathbb{K}) \right)^{S_{\gamma_0}^1}. \end{aligned}$$

Let \mathfrak{H} denote the homeomorphism in the property (iii) of the slice above. Then

$$\begin{aligned} C_q(\mathcal{L}, \gamma_0; \mathbb{K}) &= H_q(\Lambda(\gamma_0) \cup \{\gamma_0\}, \Lambda(\gamma_0); \mathbb{K}) \\ &= H_q(\text{Im}(\mathfrak{H}) \cap \Lambda(\gamma_0) \cup \{\gamma_0\}, \text{Im}(\mathfrak{H}) \cap \Lambda(\gamma_0); \mathbb{K}) \\ &= H_q((-\nu, \nu) \times (\Sigma_{2\epsilon} \cap \Lambda(\gamma_0) \cup \{\gamma_0\}), (-\nu, \nu) \times (\Sigma_{2\epsilon} \cap \Lambda(\gamma_0)); \mathbb{K}) \\ &= H_q(\Sigma_{2\epsilon} \cap \Lambda(\gamma_0) \cup \{\gamma_0\}, \Sigma_{2\epsilon} \cap \Lambda(\gamma_0); \mathbb{K}) \\ &= H_q(\Sigma_{2\epsilon} \cap \mathcal{L}_c, \Sigma_{2\epsilon} \cap \mathcal{L}_c \setminus \{\gamma_0\}; \mathbb{K}). \end{aligned} \tag{7.6}$$

As before using Theorem 3.3, the excision and Corollary A.6 we can prove

$$\begin{aligned}
(7.6) &= H_q(\mathcal{L}_c \cap \overline{\Sigma_\epsilon}, \mathcal{L}_c \cap \overline{\Sigma_\epsilon} \setminus \{\gamma_0\}; \mathbb{K}) \\
&= H_q((\mathcal{L}^*)_c \cap \overline{\Sigma_\epsilon}, (\mathcal{L}^*)_c \cap \overline{\Sigma_\epsilon} \setminus \{\gamma_0\}; \mathbb{K}) \\
&= H_q((\mathcal{E}^*|_{\overline{H_\epsilon}})_c, (\mathcal{E}^*|_{\overline{H_\epsilon}})_c \setminus \{\gamma_0\}; \mathbb{K}) \\
&= H_q((\mathcal{E}^*|_{N\mathcal{O}(\epsilon)\gamma_0})_c, (\mathcal{E}^*|_{N\mathcal{O}(\epsilon)\gamma_0})_c \setminus \{\gamma_0\}; \mathbb{K}) \\
&= H_q((\mathcal{F}^*)_c, (\mathcal{F}^*)_c \setminus \{\gamma_0\}; \mathbb{K}) \\
&= H_{m^-(S^1 \cdot \gamma_0)}(\mathbf{H}^-(B)_{\gamma_0}, \mathbf{H}^-(B)_{\gamma_0} \setminus \{0\}; \mathbb{K}) \otimes C_{q-m^-(S^1 \cdot \gamma_0)}(\mathcal{L}_{\Delta\gamma_0}^\circ, 0; \mathbb{K}).
\end{aligned}$$

Hence we arrive at

$$\begin{aligned}
&C_q(\mathcal{L}, \gamma_0; \mathbb{K}) \\
&= H_{m^-(S^1 \cdot \gamma_0)}(\mathbf{H}^-(B)_{\gamma_0}, \mathbf{H}^-(B)_{\gamma_0} \setminus \{0\}; \mathbb{K}) \otimes C_{q-m^-(S^1 \cdot \gamma_0)}(\mathcal{L}_{\Delta\gamma_0}^\circ, 0; \mathbb{K}).
\end{aligned}$$

Then (7.1) follows from this and Proposition 7.1.

8 Proof of Theorem 1.11

Though there exist many repeats with the arguments in the references [3, 29, 30, 31, 32] we are also to give necessary proof details for the reader's conveniences.

Firstly, the key Lemmas 1,2 in [25] also hold true for closed geodesics on (M, F) (cf. [42, §4.2, §7.1]), that is,

Lemma 8.1 *For a closed geodesic γ on (M, F) one has:*

- (a) *Either $m^-(\gamma_k) = 0$ for all k or there exist numbers $a > 0$, $b > 0$ such that $m^-(\gamma_{k+l}) - m^-(\gamma_k) \geq la - b$ for all k, l ;*
- (b) *There are positive integers k_1, \dots, k_s and sequence $n_{ji} \in \mathbb{N}$, $i > 0$, $j = 1, \dots, s$, such that the numbers $n_{ji}k_j$ are mutually distinct, $n_{j1} = 1$, $\{n_{ji}k_j\} = \mathbb{N}$, and $m^0(\gamma_{n_{ji}k_j}) = m^0(\gamma_{k_j})$.*

Lemma 8.2 ([3, Lem.1]) *Let (X, A) be a pair of topological spaces and α a singular relative p -cycle of (X, A) . Let Σ denote the set of singular simplices of α together with all their faces. Suppose to every $\sigma \in \Sigma$, $\sigma : \Delta^q \rightarrow X$, $0 \leq q \leq p$, there is assigned a map $P(\sigma) : \Delta^q \times [0, 1] \rightarrow X$ such that*

- (i) $P(\sigma)(z, 0) = \sigma(z)$ for $z \in \Delta^q$,
- (ii) $P(\sigma)(z, t) = \sigma(z)$ if $\sigma(\Delta^q) \subset A$,
- (iii) $P(\sigma)(\Delta^q \times \{1\}) \subset A$,
- (iv) $P(\sigma) \circ (e_q^i \times id) = P(\sigma \circ e_q^i)$ for $0 \leq i \leq q$.

Then the homology class $[\alpha] \in H_p(X, A)$ vanishes. Here Δ^q is the standard q -simplex and $e_q^i : \Delta^{q-1} \rightarrow \Delta^q$ denotes the standard linear map onto the i -th face of Δ^q .

Lemma 8.3 *Let $0 \leq a < \infty$ and $0 < b < \infty$. Each (relative) singular homology class $\mathfrak{A} \in H_*(\mathring{\mathcal{L}}_b, \mathring{\mathcal{L}}_a; \mathbb{Z})$ has a smooth relative cycle representative.*

This can be proved by either finite-dimensional approximations of Λ or the axioms for homology theories of Eilenberg and Steenrod as done in [29].

For every integer $q \geq 0$ let Δ^q be the standard q -simplex, i.e., $\Delta^q = \langle e_0, \dots, e_q \rangle$, where $e_0 = 0$ and e_1, \dots, e_q is the standard basis of \mathbb{R}^q . Replacing [3, Th.1] we need the following analogue of it, whose corresponding version in the content of the Lagrangian Conley conjecture was firstly proved by Y. Long [29, Prop. 5.1] for the finite energy homology on $M = T^n$, and then by the author [31, Prop. 5.6] for the singular homology on closed manifolds.

Proposition 8.4 *Let $\mathring{\mathcal{L}}_d = \{\mathcal{L} < d\}$ for $d > 0$. Then for a piecewise C^1 -smooth q -simplex $\eta : (\Delta^q, \partial\Delta^q) \rightarrow (\Lambda M, \mathring{\mathcal{L}}_d)$, there exists an integer $m(\eta) > 0$ such that for every integer $m \geq m(\eta)$, the q -simplex*

$$\eta^m \equiv \varphi_m(\eta) : (\Delta^q, \partial\Delta^q) \rightarrow (\Lambda M, \mathring{\mathcal{L}}_{m^2d})$$

is piecewise C^1 homotopic to a (piecewise C^1) singular q -simplex

$$\eta_m : (\Delta^q, \partial\Delta^q) \rightarrow (\mathring{\mathcal{L}}_{m^2d}, \mathring{\mathcal{L}}_{m^2d})$$

with $\eta^m = \eta_m$ on $\partial\Delta^q$ and the homotopy fixes $\eta^m|_{\partial\Delta^q}$.

This can be proved by almost repeating the proof of [31, Prop.5.6]. For completeness we shall outline its proof at the end of this section.

Now we begin with proof of Theorem 1.11. By indirect arguments we make

Assumption F: There is only a finite number of distinct closed geodesics.

Then each critical orbit of \mathcal{L} is isolated. Since $0 \leq m^0(S^1 \cdot \gamma) \leq 2n - 1$ for each closed geodesic γ , from Theorem 1.7 it follows that $H_q(\Lambda(\gamma) \cup S^1 \cdot \gamma, \Lambda(\gamma); \mathbb{K}) = C_q(\mathcal{L}, S^1 \cdot \gamma; \mathbb{K}) \neq 0$ and $m^-(\gamma) = 0$ imply $q \in [0, 2n - 1]$. Moreover, by the assumption there exist a closed geodesics $\bar{\gamma}$ and an integer $\bar{p} \geq 2$ such that $m^-(\bar{\gamma}^k) \equiv 0$ and $H_{\bar{p}}(\Lambda(\bar{\gamma}) \cup S^1 \cdot \bar{\gamma}, \Lambda(\bar{\gamma}); \mathbb{K}) \neq 0$. Hence we can find an integer $p \geq \bar{p}$ and a closed geodesic γ_0 such that

$$\left. \begin{aligned} m^-(\gamma_0^k) &\equiv 0, & H_p(\Lambda(\gamma_0) \cup S^1 \cdot \gamma_0, \Lambda(\gamma_0); \mathbb{K}) &\neq 0, \\ H_q(\Lambda(\gamma) \cup S^1 \cdot \gamma, \Lambda(\gamma); \mathbb{K}) &= 0 & \forall q > p \\ \text{for each closed geodesic } \gamma &\text{ with } m^-(\gamma^k) &\equiv 0. \end{aligned} \right\} \quad (8.1)$$

By Lemma 8.1(a) we can find $A > 0$ such that every closed geodesics γ with $\mathcal{L}(\gamma) > A$ either satisfies $m^-(\gamma) > p + 1$ or $m^-(\gamma^k) \equiv 0$. From this, (8.1) and Theorem 1.7 it follows that

$$H_{p+1}(\Lambda(\gamma) \cup S^1 \cdot \gamma, \Lambda(\gamma); \mathbb{K}) = 0 \quad (8.2)$$

for every closed geodesic γ with $\mathcal{L}(\gamma) > A$. We conclude

$$H_{p+1}(\Lambda, \Lambda(\gamma_0^m) \cup S^1 \cdot \gamma_0^m; \mathbb{K}) = 0 \quad \text{if } \mathcal{L}(\gamma_0^m) > A. \quad (8.3)$$

Otherwise, take $0 \neq \mathfrak{B} \in H_{p+1}(\Lambda, \Lambda(\gamma_0^m) \cup S^1 \cdot \gamma_0^m; \mathbb{K})$ and a singular cycle Z of $(\Lambda, \Lambda(\gamma_0^m) \cup S^1 \cdot \gamma_0^m)$ which represents \mathfrak{B} . Since the support of Z is compact we can

choose a large regular value $b > \mathcal{L}(\gamma_0^m) = m^2 \mathcal{L}(\gamma_0)$ so that $\text{supp}(Z) \subset \mathring{\mathcal{L}}_b$. Clearly, Z cannot be homologous to zero in $(\mathring{\mathcal{L}}_b, \Lambda(\gamma_0^m) \cup S^1 \cdot \gamma_0^m)$ (otherwise it is homologous to zero in $(\Lambda, \Lambda(\gamma_0^m) \cup S^1 \cdot \gamma_0^m)$.) Hence

$$H_{p+1}(\mathring{\mathcal{L}}_b, \Lambda(\gamma_0^m) \cup S^1 \cdot \gamma_0^m; \mathbb{K}) \neq 0.$$

As showed in the proof of [3, Theorem 3], under Assumption F the standard Morse theoretic arguments yield a closed geodesic γ' such that $b > \mathcal{L}(\gamma') \geq \mathcal{L}(\gamma_0^m) > A$ and

$$H_{p+1}(\Lambda(\gamma') \cup S^1 \cdot \gamma', \Lambda(\gamma'); \mathbb{K}) \neq 0.$$

This contradicts to (8.2). (8.3) is proved.

Applying Lemma 8.1(b) to γ_0 we get integers $k_i \geq 2$, $i = 1, \dots, s$, such that

$$m^0(\gamma_0) = m^0(\gamma_0^k) \quad \forall k \in \mathbb{N} \setminus \cup_{i=1}^s k_i \mathbb{N}. \quad (8.4)$$

As stated at the end of the proof of [3, Theorem 3], using (8.3) and the exact sequence for the triple $(\Lambda, \Lambda(c^m) \cup S^1 \cdot c^m, \Lambda(c^m))$,

$$\begin{aligned} \rightarrow H_{p+1}(\Lambda(\gamma_0^m) \cup S^1 \cdot \gamma_0^m, \Lambda(\gamma_0^m); \mathbb{K}) \rightarrow H_{p+1}(\Lambda, \Lambda(\gamma_0^m); \mathbb{K}) \rightarrow H_{p+1}(\Lambda, \Lambda(\gamma_0^m) \cup S^1 \cdot \gamma_0^m; \mathbb{K}) \\ \rightarrow H_p(\Lambda(\gamma_0^m) \cup S^1 \cdot \gamma_0^m, \Lambda(\gamma_0^m); \mathbb{K}) \xrightarrow{i_*} H_p(\Lambda, \Lambda(\gamma_0^m); \mathbb{K}) \rightarrow H_p(\Lambda, \Lambda(\gamma_0^m) \cup S^1 \cdot \gamma_0^m; \mathbb{K}) \rightarrow \end{aligned}$$

we obtain that for each integer m with $\mathcal{L}(\gamma_0^m) > A$,

$$i_* : H_p(\Lambda(\gamma_0^m) \cup S^1 \cdot \gamma_0^m, \Lambda(\gamma_0^m); \mathbb{K}) \rightarrow H_p(\Lambda, \Lambda(\gamma_0^m); \mathbb{K})$$

is injective. This, (8.4) and Theorem 1.9 lead to the claim on page 385 of [3].

Claim 8.5 *For any $m \in \mathbb{N} \setminus \cup_{i=1}^s k_i \mathbb{N}$ with $\mathcal{L}(\gamma_0^m) > A$, the composition*

$$H_p(\Lambda(\gamma_0) \cup S^1 \cdot \gamma_0, \Lambda(\gamma_0); \mathbb{K}) \xrightarrow{\varphi_{m*}} H_p(\Lambda(\gamma_0^m) \cup S^1 \cdot \gamma_0^m, \Lambda(\gamma_0^m); \mathbb{K}) \xrightarrow{i_*} H_p(\Lambda, \Lambda(\gamma_0^m); \mathbb{K})$$

is injective.

Hence as in the proof of [3, Theorem 3] we shall be able to complete the proof of Theorem 1.11 provided that we may prove that the homomorphism φ_{m*} induced by

$$\varphi_m : (\Lambda(\gamma_0) \cup S^1 \cdot \gamma_0, \Lambda(\gamma_0)) \rightarrow (\Lambda, \Lambda(\gamma_0^m))$$

between their singular homology groups maps some nonzero class $\mathfrak{C} \in H_p(\Lambda(\gamma_0) \cup S^1 \cdot \gamma_0, \Lambda(\gamma_0); \mathbb{K})$ to the zero in $H_p(\Lambda, \Lambda(\gamma_0^m); \mathbb{K})$ for some $m \in \mathbb{N} \setminus \cup_{i=1}^s k_i \mathbb{N}$ with $\mathcal{L}(\gamma_0^m) > A$. However, for such a nonzero class $\mathfrak{C} \in H_p(\Lambda(\gamma_0) \cup S^1 \cdot \gamma_0, \Lambda(\gamma_0); \mathbb{K})$ it is impossible to find a piecewise C^1 relative cycle representative. Fortunately, for each $\mathcal{L}(\gamma_0) < b \leq \infty$ we have the following commutative diagram

$$\begin{array}{ccc} (\Lambda(\gamma_0) \cup S^1 \cdot \gamma_0, \Lambda(\gamma_0)) & \xrightarrow{\varphi_m} & (\Lambda, \Lambda(\gamma_0^m)), \\ \downarrow \iota & \nearrow \varphi_m & \\ (\mathring{\mathcal{L}}_b, \Lambda(\gamma_0)) & & \end{array}$$

where ι denotes the inclusion and $\mathring{\mathcal{L}}_b = \{\mathcal{L} < b\}$ as before. So it suffices to prove $\varphi_{m*}(\iota_*(\mathfrak{C}))$ vanishing in $H_p(\mathring{\mathcal{L}}_b, \Lambda(\gamma_0^m); \mathbb{K})$ for some $m \in \mathbb{N} \setminus \cup_{i=1}^s k_i \mathbb{N}$ with $\mathcal{L}(\gamma_0^m) > A$.

Since γ_0 is not a local minimum of the functional \mathcal{L} , $\Lambda(\gamma_0) \neq \emptyset$ and thus we can choose a path connected neighborhood \mathcal{U} of $S^1 \cdot \gamma_0$ such that each point of \mathcal{U} can be connected to a point of $\mathcal{U} \cap \Lambda(\gamma_0)$ by a smooth path in \mathcal{U} . (Indeed, let $\mathcal{O} = S^1 \cdot \gamma_0$ and we may take $\mathcal{U} = \mathcal{N}(\mathcal{O}, \varepsilon) = \text{EXP}(N\mathcal{O})$ as below (1.12). Then for $x \in \mathcal{O}$, $0_x \in N\mathcal{O}(\varepsilon)_x$ is not a local minimum of the restriction of $\mathcal{F} = \mathcal{L} \circ (\text{EXP}|_{N\mathcal{O}(\varepsilon)})$ to the fiber $N\mathcal{O}(\varepsilon)_x$. The desired claim follows from this and the convexity of $N\mathcal{O}(\varepsilon)_x$ immediately.)

Let $d = \mathcal{L}(\gamma_0)$, and fix a $b > d$. By Lemma 8.3 we may take a smooth relative cycle representative σ of $\iota_*(\mathfrak{C}) \in H_p(\mathring{\mathcal{L}}_b, \Lambda(\gamma_0); \mathbb{K})$ in $(\mathring{\mathcal{L}}_b, \Lambda(\gamma_0))$. We can also require that the range (or carrier) of σ is contained in the given neighborhood \mathcal{U} of $S^1 \cdot \gamma_0$.

Denote by $\Sigma(\sigma)$ the set of all p -simplices of σ together with all their faces contained in σ . and by $\Sigma_j(\sigma) = \{\mu \in \Sigma(\sigma) \mid \dim \mu \leq j\}$ for $0 \leq j \leq p$. (Recall $p \geq \bar{p} \geq 2$). Let $K = \{k_1, \dots, k_s\}$, and so $\mathbb{N} \setminus \cup_{i=1}^s k_i \mathbb{N} = \mathbb{N} \setminus NK$. Let m_0 be the smallest integer such that $\mathcal{L}(\gamma_0^m) > A$.

Proposition 8.6 *There exists an integer $m \in \mathbb{N} \setminus K\mathbb{N}$ with $m > m_0$ such that for every $\mu \in \Sigma(\sigma)$ with $\mu : \Delta^r \rightarrow \Lambda$ and $0 \leq r \leq p$, there exists a homotopy $P(\varphi_m(\mu)) : \Delta^r \times [0, 1] \rightarrow \Lambda$ such that the properties (i) to (iv) in Lemma 8.2 hold for $(X, A) = (\Lambda, \Lambda(\gamma_0^m))$.*

Hence $\varphi_{m*}(\iota_*(\mathfrak{C}))$ vanishes in $H_p(\Lambda, \Lambda(\gamma_0^m); \mathbb{K})$ by Lemma 8.2. This contradicts to Claim 8.5, and therefore the proof of Theorem 1.11 is completed once we prove Proposition 8.6.

As in the proof of [29, Prop. 5.2] we can use Proposition 8.4, Lemma 8.2 and [31, Lemma A.4] to prove Proposition 8.6 as follows.

Proof of Proposition 8.6. Now $\Sigma_0(\sigma)$ consists of finite points in \mathcal{U} . For a given $\mu \in \Sigma_0(\sigma)$, if $\mu \in \Sigma_0(\sigma)$ sits in $\Lambda(\gamma_0) \cap \mathcal{U}$ we define the homotopy $P(\mu) : \Delta^0 \times [0, 1] \rightarrow \mathcal{U}$ by $P(\mu)(\mathbf{0}, t) = \mu \ \forall t \in [0, 1]$; if $\mu \in \Sigma_0(\sigma)$ is not in $\Lambda(\gamma_0) \cap \mathcal{U}$ we fix a smooth homotopy $P(\mu) : \Delta^0 \times [0, 1] \rightarrow \mathcal{U}$ given by a smooth path from μ to a point of $\Lambda(\gamma_0) \cap \mathcal{U}$. Since $\Sigma_j(\sigma^m) = \varphi_m(\Sigma_j(\sigma))$ we define $P(\mu^m) = \varphi_m \circ P(\mu)$ for each $m \in \mathbb{N}$. Clearly, these homotopies satisfy properties (i) to (iv) in Lemma 8.2 for $(X, A) = (\Lambda, \Lambda(\gamma_0^m))$. We can choose m to be any integer $m \in \mathbb{N} \setminus K\mathbb{N}$ with $m > m_0$.

Fix an integer $0 < r \leq p$. Suppose that we have found an integer $k \in \mathbb{N} \setminus K\mathbb{N}$ with $k > m_0$ such that for every $\mu \in \Sigma_{r-1}(\sigma)$ there exists a piecewise C^1 homotopy $P(\varphi_k(\mu)) : \Delta^{r-1} \times [0, 1] \rightarrow \Lambda$ such that the properties (i) to (iv) in Lemma 8.2 hold for $(X, A) = (\Lambda, \Lambda(\gamma_0^k))$. For technical reasons we reparametrize each homotopy $P(\varphi_k(\mu))$ so that

$$P(\varphi_k(\mu))(\cdot, t) = P(\varphi_k(\mu))(\cdot, 1/2) \quad \forall t \in [1/2, 1]. \quad (8.5)$$

Let $\mu \in \Sigma_r(\sigma)$. If $\mu(\Delta^r) \subset \Lambda(\gamma_0)$ we set $P(\varphi_k(\mu)) = \varphi_k \circ P(\mu)$. If $\mu(\Delta^r) \not\subset \Lambda(\gamma_0)$ (and so $\varphi_k(\mu)(\Delta^r) \not\subset \Lambda(\gamma_0^k)$), the property (iv) shows that the piecewise C^1 maps

$P(\varphi_k \circ \mu \circ e_r^i), 0 \leq i \leq r$, induce a piecewise C^1 map $Q' : \partial\Delta^r \times [0, 1] \rightarrow \Lambda$ such that

$$Q' \circ (e_r^i \times id) = P(\varphi_k \circ \mu \circ e_r^i), \quad i = 0, \dots, r. \quad (8.6)$$

For $z \in \partial\Delta^r$ let $z = e_r^i(u)$, where $u \in \Delta^{r-1}$. (8.5)-(8.6) imply

$$\begin{aligned} Q'(z, 0) &= Q' \circ (e_r^i \times id)(u, 0) = P(\varphi_k \circ \mu \circ e_r^i)(u, 0) \\ &= \varphi_k \circ \mu \circ e_r^i(u, 0) = \varphi_k \circ \mu(z), \\ Q'(z, 1/2) &= P(\varphi_k \circ \mu \circ e_r^i)(u, 1/2) = P(\varphi_k \circ \mu \circ e_r^i)(u, 1) \subset \Lambda(\gamma_0^k). \end{aligned}$$

Since Λ is a smooth Hilbert manifold and $\Delta^r \times [0, 1]$ can retract to $\partial\Delta^r \times [0, 1]$ through a piece smooth strong retracting deformation we can take a piecewise C^1 homotopy extension of the map $Q', Q : \Delta^r \times [0, 1] \rightarrow \Lambda$, such that $Q(\cdot, 0) = Q'(\cdot, 0) = \varphi_k \circ \mu$. This leads to a piecewise C^1 singular r -simplex

$$Q(\cdot, 1/2) : (\Delta^r, \partial\Delta^r) \rightarrow (\Lambda, \Lambda(\gamma_0^k))$$

because $Q(z, 1/2) = Q'(z, 1/2) \in \Lambda(\gamma_0^k)$ for all $z \in \partial\Delta^r$. By Proposition 8.4 there exists an integer $\hat{m}_r(\mu) > 0$ such that for each integer $l \geq \hat{m}_r(\mu)$ there exists a piecewise C^1 -homotopy $H : \Delta^r \times [\frac{1}{2}, 1] \rightarrow \Lambda$ such that

$$H(\cdot, 1/2) = \varphi_l \circ Q(\cdot, 1/2), \quad (8.7)$$

$$H(z, t) = \varphi_l \circ Q(\cdot, 1/2)(z) \quad \forall z \in \partial\Delta^r \text{ \& } s \in [1/2, 1], \quad (8.8)$$

$$H(\cdot, 1) : (\Delta^r, \partial\Delta^r) \rightarrow (\Lambda(\gamma_0^k), \Lambda(\gamma_0^k)). \quad (8.9)$$

Take $l \in \mathbb{N}$ such that $m := lk \in \mathbb{N} \setminus K\mathbb{N}$ and that $l > \hat{m}_r(\mu)$ for all $\mu \in \Sigma_r(\sigma)$ with $\mu(\Delta^r) \not\subseteq \Lambda(\gamma_0)$. For such an integer m the piecewise C^1 homotopy associated with $\mu \in \Sigma_r(\sigma)$ can be defined as follows.

- If $\dim \mu < r$, we define $P(\varphi_m(\mu)) = \varphi_l \circ P(\varphi_k(\mu))$.
- If $\dim \mu = r$ and $\mu(\Delta^r) \subset \Lambda(\gamma_0)$ we set $P(\varphi_m(\mu))(z, t) := \varphi_m(\mu)(z)$ for $t \in [0, 1]$.
- If $\dim \mu = r$ and $\mu(\Delta^r) \not\subseteq \Lambda(\gamma_0)$, let $Q = Q(\mu)$ and $H = H(\mu)$ be the homotopies as above, and set

$$P(\varphi_m(\mu))(z, t) := \begin{cases} \varphi_l \circ Q(\mu)(z, t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ H(\mu)(z, t) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

By (8.7)-(8.9) it is easily checked that P satisfies the properties (i)-(iv) in Lemma 8.2 for $(X, A) := (\Lambda, \Lambda(\gamma_0^m))$. This completes the proof of Proposition 8.6 is proved. \square

Proof of Proposition 8.4. Since Δ^q and $\partial\Delta^q$ are compact,

$$\mathfrak{K}_0(\eta) := \max \{ \mathcal{L}(\eta(x)) \mid x \in \partial\Delta^q \} \quad \text{and} \quad \mathfrak{K}_1(\eta) := \max \{ \mathcal{L}(\eta(x)) \mid x \in \Delta^q \}$$

are always finite. Following [29], for $t, s \in [0, 1]$ let $e(t, s) = (t, \dots, t, s) \in \mathbb{R}^q \times [0, 1]$ and thus the barycenter of $\Delta^q \times \{s\} \subset \mathbb{R}^q \times \{s\}$ is $\hat{e}(s) := e(1/(q+1), s)$. Set

$$\Delta^q(s) = e((1-s)/(q+1), s) + (s\Delta^q) \times \{0\} \subset \Delta^q \times \{s\}$$

for $s \in [0, 1]$. (Clearly, $\Delta^q(1) = \Delta^q \times \{1\}$ and $\Delta^q(0) = \hat{e}(0)$.) Denote by $L(s)$ the straight line passing through $e(0, s)$ and $\hat{e}(s)$ successively in $\mathbb{R}^q \times \{s\}$, that is, $L(s) = \{e(t, s) \mid t \in \mathbb{R}\}$. Then we have an orthogonal subspace decomposition $\mathbb{R}^q \times \{s\} = V_{q-1,s} \times L(s)$, and each $w \in \Delta^q \times \{s\}$ may be uniquely written as $w = (v, \tau) \in V_{q-1,s} \times L(s)$. For such a $w = (v, \tau) \in \Delta^q(s)$ denote by $l(v, s)$ the intersection segment of $\Delta^q(s)$ with the straight line passing through w and parallel to $L(s)$. Actually $l(v, s) = \{(v, e(t, s)) \mid a(v, s) \leq t \leq b(v, s)\}$, where $a(v, s)$ and $b(v, s)$ depend piecewise smoothly on (v, s) .

For $w = (v, \tau) \in \Delta^q(s)$ define a v -parametrized curve

$$\sharp \eta_v : l(v, s) \rightarrow \Lambda, \quad \tau \mapsto \eta(v, \tau) \quad \text{for } (v, \tau) = w \in [V_{q-1,s} \times L(s)] \cap \Delta^q(s),$$

which is piecewise C^1 , and the corresponding initial value curves

$$\eta_v^{\text{ini}} : l(v, s) \rightarrow M, \quad \tau \mapsto \eta(v, \tau)(a(v, s))$$

for $(v, \tau) = w \in [V_{q-1,s} \times L(s)] \cap \Delta^q(s)$, which is also piecewise C^1 as showed in Appendix 2 of [31]. (This can also be derived from [43, Lemma 8.25] which claimed: A C^1 -curve $\mathfrak{E} : [0, 1] \rightarrow H^1(S^1, M)$ considered as map $[0, 1] \times [0, 1] \rightarrow M$ by putting $\mathfrak{E}(s, t) = \mathfrak{E}(s)(t)$ is a homotopy between the end points $\mathfrak{E}(0), \mathfrak{E}(1) \in H^1(S^1, M)$. Moreover, $\frac{\partial \mathfrak{E}}{\partial s}(s, t)$ is continuous vector field on M along \mathfrak{E}). As in [29] using η_v^{ini} we define $\eta_m : \Delta^q \rightarrow \Lambda$ by $\eta_m(w) = (\sharp \eta_v)_m(t)$, where (v, t) is the unique $V_{q-1,1} \times L(1)$ decomposition of $(w, 1) \in \Delta^q(1) = \Delta^q \times \{1\}$. Then η_m is piecewise C^1 . Writing $(w, s) \in \Delta^q \times [0, 1]$ as $(v, t, s) \in [V_{q-1,s} \times L(s)] \times [0, 1]$ we define a homotopy $H : \Delta^q \times [0, 1] \rightarrow \Lambda$ by

$$H(w, s) = \begin{cases} \eta^m(w) & \text{if } (w, s) \in (\Delta^q \times \{s\}) \setminus \Delta^q(s), \\ (\sharp \eta_v)_m(t) & \text{if } (w, s) = (v, t, s) \in \Delta^q(s). \end{cases}$$

It is piecewise C^1 and satisfies $H(w, 0) = \eta^m$, $H(w, 1) = \eta_m$ and $H(w, s) = \eta^m(w)$ for any $(w, s) \in \partial \Delta^q \times [0, 1]$. Now for $(w, s) \in (\Delta^q \times \{s\}) \setminus \Delta^q(s)$ it holds that

$$\mathcal{L}(H(w, s)) = m^2 \mathcal{L}(\eta(w)) \leq m^2 \mathfrak{K}_1(\eta),$$

and for $(w, s) = (v, t, s) \in \Delta^q(s)$ it follows from [3, (1)] that

$$\begin{aligned} \mathcal{L}(H(w, s)) &= \mathcal{L}((\sharp \eta_v)_m(t)) \\ &\leq (m + 2|l(v, s)|)((m - 1)\kappa_0(\sharp \eta_v) + \kappa_1(\sharp \eta_v) + 2\kappa_2(\sharp \eta_v)) \\ &\leq (m + 4)((m - 1)\mathfrak{K}_0(\eta) + \mathfrak{K}_1(\eta) + \int_{a(v,s)}^{b(v,s)} \left\| \frac{d}{dt} \eta(v, t)(0) \right\|_1^2 dt) \\ &\leq (m + 4)((m - 1)\mathfrak{K}_0(\eta) + \mathfrak{K}_1(\eta) + \mathcal{C}(\eta)). \end{aligned}$$

Here $\mathcal{C}(\tilde{\eta}) := \max\{|\nabla_x(\eta(x)(0))|^2 \mid x \in \Delta^q\} < +\infty$ because $\Delta^q \ni x \rightarrow \eta(x)(0) \in M$ is piecewise C^1 . Proposition 8.4(iii)-(v) follows immediately. \square

Remark 8.7 Our proof method is slightly different from [3]. It is key for us that γ_0 is not a local minimum of the functional \mathcal{L} , which comes from (8.1) and thus our assumption that there exist a closed geodesics $\bar{\gamma}$ and an integer $\bar{p} \geq 2$ such that $m^-(\bar{\gamma}^k) \equiv 0$ and $H_{\bar{p}}(\Lambda(\bar{\gamma}) \cup S^1 \cdot \bar{\gamma}, \Lambda(\bar{\gamma}); \mathbb{K}) \neq 0$. According to [3, Theorem 3] one should make the following weaker:

Assumption: There exist a closed geodesics γ with $m^-(\gamma^k) \equiv 0 \ \forall k$ such that $H_q(\Lambda(\gamma) \cup S^1 \cdot \gamma, \Lambda(\gamma); \mathbb{K}) \neq 0$ for some $q \in \mathbb{N} \cup \{0\}$ and that γ is not an absolute minimum of \mathcal{L} in its free homotopy class.

Let us explain how such an assumption leads to the following sentence at the beginning of the proof of [3, Theorem 3, page 385] (in our notations):

Sentence ([3, lines 5-6 on page 385]): Choosing a different γ , if necessary, we can find $p \in \mathbb{N}$ such that $H_p(\Lambda(\gamma) \cup S^1 \cdot \gamma, \Lambda(\gamma); \mathbb{Z}) \neq 0$ and $H_q(\Lambda(\beta) \cup S^1 \cdot \beta, \Lambda(\beta); \mathbb{Z}) = 0$ for every $q > p$ and every closed geodesic β with $m^-(\beta^k) \equiv 0$.

In fact, by Theorem 1.8 the following case cannot occur:

$$H_0(\Lambda(\gamma) \cup S^1 \cdot \gamma, \Lambda(\gamma); \mathbb{K}) \neq 0 \quad \text{and} \quad H_k(\Lambda(\gamma) \cup S^1 \cdot \gamma, \Lambda(\gamma); \mathbb{K}) = 0 \ \forall k \in \mathbb{N}.$$

Hence we may choose $s \in \mathbb{N}$ such that $H_s(\Lambda(\gamma) \cup S^1 \cdot \gamma, \Lambda(\gamma); \mathbb{K}) \neq 0$ and

$$H_q(\Lambda(\gamma) \cup S^1 \cdot \gamma, \Lambda(\gamma); \mathbb{K}) = 0 \ \forall q > s$$

because the shifting theorem implies that $H_q(\Lambda(\alpha) \cup S^1 \cdot \alpha, \Lambda(\alpha); \mathbb{K}) = 0$ for all $q \notin [0, 2n-1]$ for every closed geodesic α with $m^-(\alpha) = 0$.

- If $H_q(\Lambda(\beta) \cup S^1 \cdot \beta, \Lambda(\beta); \mathbb{K}) = 0$ for every $q > s$ and every closed geodesic β with $m^-(\beta^k) \equiv 0$, then we may take $p = s$ in the above Sentence. In this case the closed geodesic γ is not a local minimum of \mathcal{L} if $s > 1$ by Theorem 1.8(ii). If $s = 1$ then γ is a local minimum of \mathcal{L} by Theorem 1.8(i), but is not an absolute minimum of \mathcal{L} in the free homotopy class of γ by the above Assumption.

- If there must exist an integer $l > s$ and a closed geodesic $\bar{\gamma}$ with $m^-(\bar{\gamma}^k) \equiv 0$ such that $H_l(\Lambda(\bar{\gamma}) \cup S^1 \cdot \bar{\gamma}, \Lambda(\bar{\gamma}); \mathbb{K}) \neq 0$, it is easily seen that such a pair $(l, \bar{\gamma})$ can be chosen so that $H_q(\Lambda(\beta) \cup S^1 \cdot \beta, \Lambda(\beta); \mathbb{K}) = 0$ for every integer $q > l$ and for every closed geodesic β with $m^-(\beta^k) \equiv 0$. In this case γ and p in the above Sentence can be chosen as $\bar{\gamma}$ and l , respectively. Since $p = l \geq 2$ Theorem 1.8(ii) shows that $\gamma = \bar{\gamma}$ is not a local minimum of \mathcal{L} .

Summarizing the above arguments we equivalently expressed the above Assumption as follows:

- either $H_1(\Lambda(\gamma) \cup S^1 \cdot \gamma, \Lambda(\gamma); \mathbb{K}) \neq 0$ and $H_q(\Lambda(\beta) \cup S^1 \cdot \beta, \Lambda(\beta); \mathbb{K}) = 0$ for every $q > 1$ and every closed geodesic β with $m^-(\beta^k) \equiv 0$, and γ is not an absolute minimum of \mathcal{L} in the free homotopy class of γ ;
- or $\exists p \geq 2$ and a closed geodesic γ' such that $H_p(\Lambda(\gamma') \cup S^1 \cdot \gamma', \Lambda(\gamma'); \mathbb{K}) \neq 0$ and that $H_q(\Lambda(\beta) \cup S^1 \cdot \beta, \Lambda(\beta); \mathbb{K}) = 0$ for every $q > p$ and every closed geodesic β with $m^-(\beta^k) \equiv 0$. (Hence γ' not a local minimum of \mathcal{L} .)

Hence the above Assumption may lead to the above Sentence. Comparing with the method by Bangert and Klingenberg [3] ours can only deal with the latter case.

A Appendix: The splitting theorems in [32, 33, 34, 26]

Let H be a Hilbert space with inner product $(\cdot, \cdot)_H$ and the induced norm $\|\cdot\|$, and let X be a Banach space with norm $\|\cdot\|_X$, such that

(S) $X \subset H$ is dense in H and $\|x\| \leq \|x\|_X \ \forall x \in X$.

For an open neighborhood U of the origin $0 \in H$, write $U_X = U \cap X$ as an open neighborhood of 0 in X . Let $\mathcal{L} \in C^1(U, \mathbb{R})$ have 0 as an isolated critical point. Suppose that there exist maps $A \in C^1(U_X, X)$ and $B \in C(U_X, L_s(H))$ such that

$$\mathcal{L}'(x)(u) = (A(x), u)_H \quad \forall x \in U_X \text{ and } u \in X, \quad (\text{A.1})$$

$$(A'(x)(u), v)_H = (B(x)u, v)_H \quad \forall x \in U_X \text{ and } u, v \in X. \quad (\text{A.2})$$

(These imply: (a) $\mathcal{L}|_{U_X} \in C^2(U_X, \mathbb{R})$, (c) $d^2\mathcal{L}|_{U_X}(x)(u, v) = (B(x)u, v)_H$ for any $x \in U_X$ and $u, v \in X$, (c) $B(x)(X) \subset X \ \forall x \in U_X$). Furthermore we assume B to satisfy the following properties:

- (B1) If $u \in H$ such that $B(0)(u) = v$ for some $v \in X$, then $u \in X$. Moreover, all eigenvectors of the operator $B(0)$ that correspond to negative eigenvalues belong to X .
- (B2) The map $B : U_X \rightarrow L_s(H)$ has a decomposition $B(x) = P(x) + Q(x)$ for each $x \in U_X$, where $P(x) : H \rightarrow H$ is a positive definite linear operator and $Q(x) : H \rightarrow H$ is a compact linear operator with the following properties:
 - (i) For any sequence $\{x_k\} \subset U_X$ with $\|x_k\| \rightarrow 0$ it holds that $\|P(x_k)u - P(0)u\| \rightarrow 0$ for any $u \in H$;
 - (ii) The map $Q : U \cap X \rightarrow L(H)$ is continuous at 0 with respect to the topology induced from H on $U \cap X$;
 - (iii) There exist positive constants $\eta_0 > 0$ and $C_0 > 0$ such that

$$(P(x)u, u) \geq C_0\|u\|^2 \quad \forall u \in H, \ \forall x \in X \text{ with } \|x\| < \eta_0.$$

Let H^- , H^0 and H^+ be the negative definite, null and positive definite spaces of $B(0)$. Then $H = H^- \oplus H^0 \oplus H^+$. By (B1) and (B2) both H^0 and H^- are finite-dimensional subspaces contained in X . Denote by P^* the orthogonal projections onto H^* , $*$ = +, -, 0, and by $X^* = X \cap H^* = P^*(X)$, $*$ = +, -. Then X^+ is dense in H^+ , and $(I - P^0)|_X = (P^+ + P^-)|_X : (X, \|\cdot\|_X) \rightarrow (X^+ + X^-, \|\cdot\|)$ is also continuous because all norms are equivalent on a linear space of finite dimension. These give the topological direct sum decomposition: $X = H^- \oplus H^0 \oplus X^+$. $m^0 = \dim H^0$ and $m^- = \dim H^-$ are called the **nullity** and the **Morse index** of critical point 0 of \mathcal{L} , respectively. The critical point 0 is called **nondegenerate** if $m^0 = 0$. The following is, unless (iv), Theorem 1.1 of [32], which is a special version of Theorem 2.1 in [34].

Theorem A.1 *Under the above assumptions (S) and (B1)-(B2), there exist a positive number $\epsilon \in \mathbb{R}$, a C^1 map $h : \mathbf{B}_\epsilon(H^0) \rightarrow X^+ + X^-$ satisfying $h(0) = 0$ and*

$$(I - P^0)A(z + h(z)) = 0 \quad \forall z \in \mathbf{B}_\epsilon(H^0),$$

an open neighborhood W of 0 in H and an origin-preserving homeomorphism

$$\Phi : \mathbf{B}_\epsilon(H^0) \times (\mathbf{B}_\epsilon(H^+) + \mathbf{B}_\epsilon(H^-)) \rightarrow W$$

of form $\Phi(z, u^+ + u^-) = z + h(z) + \phi_z(u^+ + u^-)$ with $\phi_z(u^+ + u^-) \in H^\pm$ such that

$$\mathcal{L} \circ \Phi(z, u^+ + u^-) = \|u^+\|^2 - \|u^-\|^2 + \mathcal{L}(z + h(z))$$

for all $(z, u^+ + u^-) \in \mathbf{B}_\epsilon(H^0) \times (\mathbf{B}_\epsilon(H^+) + \mathbf{B}_\epsilon(H^-))$, and that

$$\Phi(\mathbf{B}_\epsilon(H^0) \times (\mathbf{B}_\epsilon(H^+) \cap X + \mathbf{B}_\epsilon(H^-))) \subset X.$$

Moreover, Φ , h and the function $\mathbf{B}_\epsilon(H^0) \ni z \mapsto \mathcal{L}^\circ(z) := \mathcal{L}(z + h(z))$ also satisfy:

- (i) *For $z \in \mathbf{B}_\epsilon(H^0)$, $\Phi(z, 0) = z + h(z)$, $\phi_z(u^+ + u^-) \in H^-$ if and only if $u^+ = 0$;*
- (ii) *$h'(z) = -[(I - P^0)A'(z + h(z))|_{X^\pm}]^{-1}(I - P^0)A'(z + h(z))|_{H^0} \quad \forall z \in \mathbf{B}_\epsilon(H^0)$;*
- (iii) *\mathcal{L}° is C^2 , has an isolated critical point 0, $d^2\mathcal{L}^\circ(0) = 0$ and*

$$d\mathcal{L}^\circ(z_0)(z) = (A(z_0 + h(z_0)), z)_H \quad \forall z_0 \in \mathbf{B}_\epsilon(H^0), z \in H^0.$$

- (iv) *Φ is also a homeomorphism from $\mathbf{B}_\epsilon(H^0) \times \mathbf{B}_\epsilon(H^-)$ to $\Phi(\mathbf{B}_\epsilon(H^0) \times \mathbf{B}_\epsilon(H^-))$ even if the topology on the latter is taken as the induced one by X . (This implies that H and X induce the same topology in $\Phi(\mathbf{B}_\epsilon(H^0) \times \mathbf{B}_\epsilon(H^-))$.)*

Corollary A.2 ([32, 34]) (Shifting) *Under the assumptions of Theorem A.1, for any Abelian group \mathbb{K} it holds that $C_q(\mathcal{L}, 0; \mathbb{K}) \cong C_{q-m-}(\mathcal{L}^\circ, 0; \mathbb{K})$ for each $q = 0, 1, \dots$.*

Theorem A.3 ([34, Th.6.4]) *Under the assumptions of Theorem A.1, suppose that $\check{H} \subset H$ is a Hilbert subspace whose orthogonal complementary in H is finite-dimensional and is contained in X . Then $(\mathcal{L}|_{\check{H}}, \check{H}, \check{X})$ with $\check{X} := X \cap \check{H}$ also satisfies the assumptions of Theorem A.1 around the critical point $0 \in \check{H}$.*

Theorem A.4 ([34, Th.6.3]) *Under the assumptions of Theorem A.1, let $(\widehat{H}, \widehat{X})$ be another pair of Hilbert-Banach spaces satisfying (S), and let $J : H \rightarrow \widehat{H}$ be a Hilbert space isomorphism which can induce a Banach space isomorphism $J_X : X \rightarrow \widehat{X}$ (this means that $J(X) \subset \widehat{X}$ and $J|_X : X \rightarrow \widehat{X}$ is a Banach space isomorphism). Set $\widehat{U} = J(U)$ (and hence $\widehat{U}_{\widehat{X}} := \widehat{U} \cap \widehat{X} = J(U_X)$) and $\widehat{\mathcal{L}} : \widehat{U} \rightarrow \mathbb{R}$ by $\widehat{\mathcal{L}} = \mathcal{L} \circ J^{-1}$. Then $(\widehat{H}, \widehat{X}, \widehat{U}, \widehat{\mathcal{L}})$ satisfies the assumptions of Theorem A.1 too.*

Let us give the following special case of the splitting lemma by Ming Jiang [26] (i.e. the case $X = Y$) with our above notations.

Theorem A.5 ([26, Th.2.5]) *Let the assumptions above **(B1)** be satisfied when the neighborhood U_X therein is replaced by some ball $\mathbf{B}_\varepsilon(X) \subset X$. (These imply that the conditions (SP),³ (FN1)-(FN3) in [26, Th.2.5]) hold in $\mathbf{B}_\varepsilon(X)$). Suppose also that the first condition in **(B1)** and the following condition are satisfied:*

(CP1) *0 is either not in the spectrum $\sigma(B(0))$ or is an isolated point of $\sigma(B(0))$. (This holds if **(B2)** is true by Proposition B.2 in [33].)*

Then there exists a ball $\mathbf{B}_\delta(X) \subset \mathbf{B}_\varepsilon(X)$, an origin-preserving local homeomorphism φ from $\mathbf{B}_\delta(X)$ to an open neighborhood of 0 in U_X and a C^1 map $h : \mathbf{B}_\delta(H^0) \rightarrow X^+ + X^-$ such that

$$\mathcal{L}|_X \circ \varphi(x) = \frac{1}{2}(B(0)x^\perp, x^\perp)_H + \mathcal{L}(h(z) + z) \quad \forall x \in \mathbf{B}_\delta(X), \quad (\text{A.3})$$

where $z = P^0(x)$ and $x^\perp = (P^+ + P^-)(x)$. Moreover, the function $\mathbf{B}_\delta(X) \ni z \mapsto \mathcal{L}_X^\circ(z) := \mathcal{L}(h(z) + z)$ has the same properties as \mathcal{L}° in Theorem A.1.

If $X^- := X \cap H^- = H^-$ then $-B(0)|_{H^-} : H^- \rightarrow H^-$ is positive definite and therefore $(-B(0)|_{H^-})^{\frac{1}{2}} : H^- \rightarrow H^-$ is an isomorphism. It follows that

$$\frac{1}{2}(B(0)P^+x, P^+x) = \left(2^{-\frac{1}{2}}(-B(0)|_{H^-})^{\frac{1}{2}}P^-x, 2^{-\frac{1}{2}}(-B(0)|_{H^-})^{\frac{1}{2}}P^-x\right).$$

Consider the Banach space isomorphism $\Gamma : X \rightarrow X$ given by

$$X = X^- + X^0 + X^+ \ni x = x^- + x^0 + x^+ \rightarrow 2^{-\frac{1}{2}}(-B(0)|_{H^-})^{\frac{1}{2}}x^- + x^0 + x^+.$$

Replacing φ by $\varphi \circ \Gamma^{-1}$ and shrinking $\delta > 0$ suitably, (A.3) becomes

$$\mathcal{L}|_X \circ \varphi \circ \Gamma^{-1}(x) = \frac{1}{2}(B(0)x^+, x^+)_H - \|x^-\|^2 + \mathcal{L}(h(z) + z) \quad \forall x \in \mathbf{B}_\delta(X), \quad (\text{A.4})$$

where $z = P^0(x)$, $x^\star = P^\star x$, $\star = +, -$, and $\|x^-\|^2 = (x^-, x^-)_H$. In some applications such a form of Theorem A.5 is more convenient.

Clearly, the assumptions of Theorem A.1 are stronger than those of Theorem A.5. Under the conditions of Theorem A.1 these two theorems take the same maps h and hence \mathcal{L}_X° and \mathcal{L}° are same near $0 \in H^0$.

Following the methods of proof in [15, Th.5.1.17] we may obtain

Corollary A.6 (Shifting) *Under the assumption of Theorem A.5 suppose that $H^- \subset X$ and $\dim(H^0 + H^-) < \infty$. Then $C_q(\mathcal{L}|_{U_X}, 0; \mathbb{K}) = C_{q-m^-}(\mathcal{L}_X^\circ, 0; \mathbb{K})$ for each $q = 0, 1, \dots$, where $m^- := \dim H^-$.*

By simple arguments as in the proofs of [34, Th.6.3, Th.6.4] we may obtain the following corresponding results with Theorems A.3, A.4.

³In the original Theorem 2.5 of [26] the density of X in H is not needed.

Theorem A.7 *Under the assumptions of Theorem A.5, suppose that $\check{H} \subset H$ is a Hilbert subspace whose orthogonal complementary in H is finite-dimensional and is contained in X . Then $(\mathcal{L}|_{\check{H}}, \check{H}, \check{X}, \check{A} := P_{\check{H}} \circ A|_{\check{X}}, \check{B}(\cdot) := P_{\check{H}} B(\cdot)|_{\check{H}})$, where $\check{X} := X \cap \check{H}$ and $P_{\check{H}}$ is the orthogonal projection onto \check{H} , also satisfies the assumptions of Theorem A.5.*

Theorem A.8 *Under the assumptions of Theorem A.1, let (\hat{H}, \hat{X}) be another pair of Hilbert-Banach spaces satisfying **(S)**, and let $J : H \rightarrow \hat{H}$ be a Hilbert space isomorphism which can induce a Banach space isomorphism $J_X : X \rightarrow \hat{X}$. Then $(\hat{H}, \hat{X}, \hat{\mathcal{L}} = \mathcal{L} \circ J^{-1})$ satisfies the assumptions of Theorem A.5 too.*

Under Corollaries A.2, A.6 it holds that for any $q = 0, 1, \dots$,

$$C_q(\mathcal{L}, 0; \mathbb{K}) = C_{q-m-}(\mathcal{L}^\circ, 0; \mathbb{K}) = C_{q-m-}(\mathcal{L}_X^\circ, 0; \mathbb{K}) = C_q(\mathcal{L}_{U_X}, 0; \mathbb{K}). \quad (\text{A.5})$$

because \mathcal{L}° and \mathcal{L}_X° are same near $0 \in \mathbf{B}_\delta(X)$. Actually, a stronger result holds.

Theorem A.9 ([34, Cor.2.5]) *Under the assumptions of Theorem A.1 let $c = \mathcal{L}(0)$. For any open neighborhood W of 0 in U and a field \mathbb{F} , write $W_X = W \cap X$ as an open subset of X , then the inclusion*

$$(\mathcal{L}_c \cap W_X, \mathcal{L}_c \cap W_X \setminus \{0\}) \hookrightarrow (\mathcal{L}_c \cap W, \mathcal{L}_c \cap W \setminus \{0\}) \quad (\text{A.6})$$

induces isomorphisms

$$H_*(\mathcal{L}_c \cap W_X, \mathcal{L}_c \cap W_X \setminus \{0\}; \mathbb{F}) \cong H_*(\mathcal{L}_c \cap W, \mathcal{L}_c \cap W \setminus \{0\}; \mathbb{F}).$$

It is not hard to prove that the corresponding conclusion also holds true if we replace $(\mathcal{L}_c \cap W_X, \mathcal{L}_c \cap W_X \setminus \{0\})$ and $(\mathcal{L}_c \cap W, \mathcal{L}_c \cap W \setminus \{0\})$ in (A.6) by

$$(\mathring{\mathcal{L}}_c \cap W_X \cup \{0\}, \mathring{\mathcal{L}}_c \cap W_X) \quad \text{and} \quad (\mathring{\mathcal{L}}_c \cap W \cup \{0\}, \mathring{\mathcal{L}}_c \cap W)$$

respectively, where $\mathring{\mathcal{L}}_c = \{\mathcal{L} < c\}$.

B Appendix: Some related computations

1. Computing the gradient $\nabla^m \mathcal{L}_{\sigma^m}$ of \mathcal{L}_{σ^m} on \hat{H}_{σ^m} . We only consider the case $\sigma^m = -1$. From the following arguments ones easily see the case $\sigma^m = 1$. Since $\sigma^m = -1$ we have

$$\hat{H}_{\sigma^m} = \{(\xi_1, \dots, \xi_n) \in W_{loc}^{1,2} \mid \xi_1(t+1) = -\xi_1(t), \xi_j(t+1) = \xi_j(t), \forall t \in \mathbb{R}, j \geq 2\}.$$

For $\xi \in \mathbf{B}_{2\rho}(\hat{H}_{\sigma^m})$ we want to compute $\nabla^m \mathcal{L}_{\sigma^m}(\xi)$. Since \mathcal{L}_{σ^m} is C^{2-0} on $\mathbf{B}_{2\rho}(\hat{H}_{\sigma^m})$, by the continuity of $\nabla^m \mathcal{L}_{\sigma^m}$ we may assume that ξ is C^1 below. Write

$$D_v L_{\sigma^m}(t, \xi(t), \dot{\xi}(t)) = (\mathfrak{V}_1^\xi(t), \dots, \mathfrak{V}_n^\xi(t)), \quad (\text{B.1})$$

$$D_x L_{\sigma^m}(t, \xi(t), \dot{\xi}(t)) = (\mathfrak{X}_1^\xi(t), \dots, \mathfrak{X}_n^\xi(t)). \quad (\text{B.2})$$

They satisfy similar equalities to (5.37) and (5.38), and hence

$$\mathfrak{V}_1^\xi(t+1) = -\mathfrak{V}_1^\xi(t), \quad \mathfrak{X}_1^\xi(t+1) = -\mathfrak{X}_1^\xi(t) \quad \forall t \in \mathbb{R}, \quad (\text{B.3})$$

$$\mathfrak{V}_j^\xi(t+1) = \mathfrak{V}_j^\xi(t), \quad \mathfrak{X}_j^\xi(t+1) = \mathfrak{X}_j^\xi(t) \quad \forall t \in \mathbb{R}, \quad j \geq 2. \quad (\text{B.4})$$

Define $\mathfrak{G}_1^{m\xi} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\mathfrak{G}_1^{m\xi}(t) = \int_0^t \mathfrak{V}_1^\xi(s) ds - \frac{1}{2} \int_0^1 \mathfrak{V}_1^\xi(s) ds \quad \forall t \in \mathbb{R}. \quad (\text{B.5})$$

It is primitive function of the function $\mathfrak{V}_1^\xi(t)$, and satisfies

$$\mathfrak{G}_1^{m\xi}(t+1) = -\mathfrak{G}_1^{m\xi}(t) \quad \forall t \in \mathbb{R}. \quad (\text{B.6})$$

For each $j = 2, \dots, n$, we define

$$\mathfrak{G}_j^{m\xi}(t) = \int_0^t \left[\mathfrak{V}_j^\xi(s) - \int_0^1 \mathfrak{V}_j^\xi(\tau) d\tau \right] ds, \quad (\text{B.7})$$

which is a 1-periodic primitive function of the function

$$s \mapsto \mathfrak{V}_j^\xi(s) - \int_0^1 \mathfrak{V}_j^\xi(\tau) d\tau.$$

Obverse for $\eta \in \hat{H}_{\sigma^m}$ that

$$\begin{aligned} \int_0^1 \mathfrak{V}_1^\xi(t) \dot{\eta}_1(t) dt &= \int_0^1 \dot{\mathfrak{G}}_1^{m\xi}(t) \dot{\eta}_1(t) dt = (\mathfrak{G}_1^{m\xi}, \eta_1)_{\sigma^m} - m^2 \int_0^1 \mathfrak{G}_1^{m\xi}(t) \eta_1(t) dt, \\ \int_0^1 \mathfrak{V}_j^\xi(t) \dot{\eta}_j(t) dt &= \int_0^1 [\mathfrak{V}_j^\xi(t) - \int_0^1 \mathfrak{V}_j^\xi(s) ds] \dot{\eta}_j(t) dt = \int_0^1 \dot{\mathfrak{G}}_j^{m\xi}(t) \dot{\eta}_j(t) dt \\ &= (\mathfrak{G}_j^{m\xi}, \eta_j)_{\sigma^m} - m^2 \int_0^1 \mathfrak{G}_j^{m\xi}(t) \eta_j(t) dt, \quad j = 2, \dots, n. \end{aligned}$$

By this and (B.1)-(B.2) we obtain

$$\begin{aligned} d\mathcal{L}_{\sigma^m}(\xi)(\eta) &= \sum_{j=1}^n \int_0^1 \mathfrak{X}_j^\xi(t) \eta_j(t) dt + \sum_{j=1}^n \int_0^1 \mathfrak{V}_j^\xi(t) \dot{\eta}_j(t) dt \\ &= \sum_{j=1}^n \int_0^1 [\mathfrak{X}_j^\xi(t) - m^2 \mathfrak{G}_j^{m\xi}(t)] \eta_j(t) dt + (\mathfrak{G}^{m\xi}, \eta)_{\sigma^m}. \end{aligned} \quad (\text{B.8})$$

Since $\hat{H}_{\sigma^m} \ni \eta \mapsto \sum_{j=1}^n \int_0^1 [\mathfrak{X}_j^\xi(t) - m^2 \mathfrak{G}_j^{m\xi}(t)] \eta_j(t) dt$ is a bounded linear functional there exists a unique $\mathfrak{F}^\xi = (\mathfrak{F}_1^\xi, \dots, \mathfrak{F}_n^\xi) \in \hat{H}_{\sigma^m}$ such that

$$\sum_{j=1}^n \int_0^1 [\mathfrak{X}_j^\xi(t) - m^2 \mathfrak{G}_j^{m\xi}(t)] \eta_j(t) dt = (\mathfrak{F}^\xi, \eta)_{\sigma^m} \quad \forall \eta \in \hat{H}_{\sigma^m}. \quad (\text{B.9})$$

This and (B.8) lead to

$$\nabla^m \mathcal{L}_{\sigma^m}(\xi) = \mathfrak{G}^{m\xi} + \mathfrak{F}^\xi. \quad (\text{B.10})$$

Moreover, (B.9) also implies

$$\int_0^1 [\mathfrak{X}_1^\xi(t) - m^2 \mathfrak{G}_1^{m\xi}(t)] \eta_1(t) dt = m^2 \int_0^1 \mathfrak{F}_1^\xi(t) \eta_1(t) dt + \int_0^1 \dot{\mathfrak{F}}_1^\xi(t) \dot{\eta}_1(t) dt \quad (\text{B.11})$$

for any $W_{loc}^{1,2}$ map $\eta_1 : \mathbb{R} \rightarrow \mathbb{R}^n$ satisfying $\eta_1(t+1) = -\eta_1(t) \forall t \in \mathbb{R}$, and

$$\int_0^1 [\mathfrak{X}_j^\xi(t) - m^2 \mathfrak{G}_j^{m\xi}(t)] \eta_j(t) dt = m^2 \int_0^1 \mathfrak{F}_j^\xi(t) \eta_j(t) dt + \int_0^1 \dot{\mathfrak{F}}_j^\xi(t) \dot{\eta}_j(t) dt \quad (\text{B.12})$$

for any 1-periodic $W_{loc}^{1,2}$ map $\eta_j : \mathbb{R} \rightarrow \mathbb{R}^n$, $j = 2, \dots, n$.

The following lemma may be proved directly.

Lemma B.1 *Let m be a positive integer, and let $f \in L_{loc}^1(\mathbb{R}, \mathbb{R}^n)$ be bounded. If f is 1-periodic (resp. satisfies: $f(t+1) = -f(t) \forall t$) then the equation $x''(t) - m^2 x(t) = f(t)$ has a unique solution*

$$x(t) = -\frac{1}{2m} \int_t^\infty e^{m(t-s)} f(s) ds - \frac{1}{2m} \int_{-\infty}^t e^{m(s-t)} f(s) ds$$

which is 1-periodic (resp. satisfies: $x(t+1) = -x(t) \forall t$).

From (B.1)-(B.7), Lemma B.1 and (B.11)-(B.12) we derive that

$$\begin{aligned} \mathfrak{F}_j(t) &= \frac{1}{2m} \int_t^\infty e^{m(t-s)} [\mathfrak{X}_j^\xi(s) - m^2 \mathfrak{G}_j^{m\xi}(s)] ds \\ &+ \frac{1}{2m} \int_{-\infty}^t e^{m(s-t)} [\mathfrak{X}_j^\xi(s) - m^2 \mathfrak{G}_j^{m\xi}(s)] ds \end{aligned} \quad (\text{B.13})$$

for $j = 1, \dots, n$. This and (B.10) lead to

$$\begin{aligned} \nabla^m \mathcal{L}_{\sigma^m}(\xi)(t) &= \mathfrak{G}^{m\xi}(t) + \frac{1}{2m} \int_t^\infty e^{m(t-s)} \left[D_x L_{\sigma^m}(s, \xi(s), \dot{\xi}(s)) - m^2 \mathfrak{G}^{m\xi}(s) \right] ds \\ &+ \frac{1}{2m} \int_{-\infty}^t e^{m(s-t)} \left[D_x L_{\sigma^m}(s, \xi(s), \dot{\xi}(s)) - m^2 \mathfrak{G}^{m\xi}(s) \right] ds, \end{aligned} \quad (\text{B.14})$$

where by (B.5) and (B.7) $\mathfrak{G}^{m\xi} = (\mathfrak{G}_1^{m\xi}, \dots, \mathfrak{G}_n^{m\xi})$ are given by

$$\begin{aligned} \mathfrak{G}^{m\xi}(t) &= \int_0^t D_v L_{\sigma^m}(s, \xi(s), \dot{\xi}(s)) ds \\ &- \left(\int_0^1 D_v L_{\sigma^m}(s, \xi(s), \dot{\xi}(s)) ds \right) \text{diag}(1/2, t, \dots, t). \end{aligned} \quad (\text{B.15})$$

Clearly, when $\sigma^m = 1$, i.e., m is even, the above proof shows that $\nabla^m \mathcal{L}_{\sigma^m}(\xi)$ is still given by (B.14) with

$$\mathfrak{G}^{m\xi}(t) = \int_0^t D_v L_{\sigma^m}(s, \xi(s), \dot{\xi}(s)) ds - t \int_0^1 D_v \tilde{L}_{\sigma^m}(s, \xi(s), \dot{\xi}(s)) ds. \quad (\text{B.16})$$

Obverse that (B.15) and (B.16) can be written as a united form:

$$\begin{aligned}\mathfrak{G}^{m\xi}(t) &= - \left(\int_0^1 D_v L_{\sigma^m}(s, \xi(s), \dot{\xi}(s)) ds \right) \text{diag} \left(\frac{2t\sigma^m + 2t + 1 - \sigma^m}{4}, t, \dots, t \right) \\ &\quad + \int_0^t D_v L_{\sigma^m}(s, \xi(s), \dot{\xi}(s)) ds.\end{aligned}\tag{B.17}$$

Since $\hat{H}_\sigma = H_\sigma$ we get

Corollary B.2 *The gradient of \mathcal{L}_σ on $\mathbf{B}_{2\rho}(H_\sigma)$ is given by*

$$\begin{aligned}\nabla \mathcal{L}_\sigma(\xi)(t) &= \nabla^1 \mathcal{L}_\sigma(\xi)(t) \\ &= \frac{1}{2} \int_t^\infty e^{t-s} \left[D_x L_\sigma(s, \xi(s), \dot{\xi}(s)) - \mathfrak{G}^{1\xi}(s) \right] ds \\ &\quad + \frac{1}{2} \int_{-\infty}^t e^{s-t} \left[D_x L_\sigma(s, \xi(s), \dot{\xi}(s)) - \mathfrak{G}^{1\xi}(s) \right] ds + \mathfrak{G}^{1\xi}(t),\end{aligned}\tag{B.18}$$

where

$$\begin{aligned}\mathfrak{G}^{1\xi}(t) &= - \left(\int_0^1 D_v L_\sigma(s, \xi(s), \dot{\xi}(s)) ds \right) \text{diag} \left(\frac{2t\sigma + 2t + 1 - \sigma}{4}, t, \dots, t \right) \\ &\quad + \int_0^t D_v L_\sigma(s, \xi(s), \dot{\xi}(s)) ds.\end{aligned}\tag{B.19}$$

2. Computing $\nabla^m \mathcal{L}_{\sigma^m}(\xi^m)$ for $\xi^m \in \tilde{\varphi}_m(\mathbf{B}_{2\rho}(H_\sigma)) \subset \mathbf{B}_{2\sqrt{m}\rho}(\hat{H}_{\sigma^m})$. By (6.6) we may derive $D_v L_{\sigma^m}(t, x, v) = m D_v L_\sigma(mt, x, v/m)$ and hence

$$D_v L_{\sigma^m}(t, \xi^m(t), \dot{\xi}^m(t)) = m D_v L_\sigma(mt, \xi(mt), \dot{\xi}(mt)).$$

By this, (B.17) and changes of variables we obtain

$$\begin{aligned}\mathfrak{G}^{m\xi^m}(t) &= - \left(\int_0^1 D_v L_{\sigma^m}(s, \xi^m(s), (\xi^m)^\cdot(s)) ds \right) \text{diag} \left(\frac{2t\sigma^m + 2t + 1 - \sigma^m}{4}, t, \dots, t \right) \\ &\quad + \int_0^t D_v L_{\sigma^m}(s, \xi^m(s), (\xi^m)^\cdot(s)) ds \\ &= - \left(\int_0^m D_v L_\sigma(s, \xi(s), \dot{\xi}(s)) ds \right) \text{diag} \left(\frac{2t\sigma^m + 2t + 1 - \sigma^m}{4}, t, \dots, t \right) \\ &\quad + \int_0^{mt} D_v L_\sigma(s, \xi(s), \dot{\xi}(s)) ds.\end{aligned}\tag{B.20}$$

Similarly, (6.6) also leads to $D_x L_{\sigma^m}(t, x, v) = m^2 D_x L_\sigma(mt, x, v/m)$ and so

$$D_x L_{\sigma^m}(t, \xi^m(t), (\xi^m)^\cdot(t)) = m^2 D_x L_\sigma(mt, \xi(mt), \dot{\xi}(mt)).$$

Then using this and (B.14) we deduce

$$\begin{aligned}
& \nabla^m \mathcal{L}_{\sigma^m}(\xi^m)(t) = \mathfrak{G}^{m\xi^m}(t) + \\
& + \frac{1}{2m} \int_t^\infty e^{m(t-s)} \left[D_x L_{\sigma^m}(s, \xi^m(s), (\xi^m)'(s)) - m^2 \mathfrak{G}^{m\xi^m}(s) \right] ds + \\
& + \frac{1}{2m} \int_{-\infty}^t e^{m(s-t)} \left[D_x L_{\sigma^m}(s, \xi^m(s), (\xi^m)'(s)) - m^2 \mathfrak{G}^{m\xi^m}(s) \right] ds \\
& = \frac{1}{2m} \int_t^\infty e^{m(t-s)} \left[m^2 D_x L_\sigma(ms, \xi(ms), \dot{\xi}(ms)) - m^2 \mathfrak{G}^{m\xi^m}(s) \right] ds \\
& + \frac{1}{2m} \int_{-\infty}^t e^{m(s-t)} \left[m^2 D_x L_\sigma(ms, \xi(ms), \dot{\xi}(ms)) - m^2 \mathfrak{G}^{m\xi^m}(s) \right] ds + \mathfrak{G}^{m\xi^m}(t) \\
& = \frac{1}{2} \int_{mt}^\infty e^{mt-s} \left[D_x L_\sigma(s, \xi(s), \dot{\xi}(s)) - \mathfrak{G}^{m\xi^m}(s/m) \right] ds \\
& + \frac{1}{2} \int_{-\infty}^{mt} e^{s-mt} \left[D_x L_\sigma(s, \xi(s), \dot{\xi}(s)) - \mathfrak{G}^{m\xi^m}(s/m) \right] ds + \mathfrak{G}^{m\xi^m}(t). \tag{B.21}
\end{aligned}$$

Claim B.3 $\nabla^m \mathcal{L}_{\sigma^m}(\xi^m) = \tilde{\varphi}_{m\sigma}(\nabla \mathcal{L}_\sigma(\xi)) \quad \forall \xi \in \mathbf{B}_{2\rho}(H_\sigma).$

Proof. For convenience let us write

$$\begin{aligned}
D_x L_\sigma(s, \xi(s), \dot{\xi}(s)) &= (\mathfrak{Y}_1(s), \dots, \mathfrak{Y}_n(s)), \\
D_v L_\sigma(s, \xi(s), \dot{\xi}(s)) &= (\mathfrak{Z}_1(s), \dots, \mathfrak{Z}_n(s))
\end{aligned}$$

and $\mathbf{I}_j(t)$ th-j component of $\nabla^m \mathcal{L}_{\sigma^m}(\xi^m)(t) - \tilde{\varphi}_{m\sigma}(\nabla \mathcal{L}_\sigma(\xi))(t)$, $j = 1, \dots, n$. Since both $\mathfrak{Y}_j(s)$ and $\mathfrak{Z}_j(s)$ are 1-periodic for $j = 2, \dots, n$. By (B.19) we have

$$\begin{aligned}
\mathfrak{G}_j^{1\xi}(mt) &= -mt \int_0^1 \mathfrak{Z}_j(s) ds + \int_0^{mt} \mathfrak{Z}_j(s) ds \\
&= -t \int_0^m \mathfrak{Z}_j(s) ds + \int_0^{mt} \mathfrak{Z}_j(s) ds = \mathfrak{G}_j^{m\xi^m}(t)
\end{aligned}$$

for $j = 2, \dots, n$. Let $[\tilde{\varphi}_{m\sigma}(\nabla \mathcal{L}_\sigma(\xi))]_j$ be the th-j component of $\tilde{\varphi}_{m\sigma}(\nabla \mathcal{L}_\sigma(\xi))$. Then

$$\begin{aligned}
[\tilde{\varphi}_{m\sigma}(\nabla \mathcal{L}_\sigma(\xi))]_j(t) &= \frac{1}{2} \int_t^\infty e^{mt-s} \left[\mathfrak{Y}_j(s) - \mathfrak{G}_j^{1\xi}(s) \right] ds \\
&+ \frac{1}{2} \int_{-\infty}^t e^{s-mt} \left[\mathfrak{Y}_j(s) - \mathfrak{G}_j^{1\xi}(s) \right] ds + \mathfrak{G}_j^{1\xi}(mt)
\end{aligned}$$

for $j = 2, \dots, n$ by (B.18). It follows from these and (B.21) that for $2 \leq j \leq n$,

$$\begin{aligned}
\mathbf{I}_j(t) &= \frac{1}{2} \int_{mt}^\infty e^{mt-s} \left[\mathfrak{Y}_j(s) - \mathfrak{G}_j^{m\xi^m}(s/m) \right] ds \\
&+ \frac{1}{2} \int_{-\infty}^{mt} e^{s-mt} \left[\mathfrak{Y}_j(s) - \mathfrak{G}_j^{m\xi^m}(s/m) \right] ds \\
&- \frac{1}{2} \int_{mt}^\infty e^{mt-s} \left[\mathfrak{Y}_j(s) - \mathfrak{G}_j^{1\xi}(s) \right] ds \\
&- \frac{1}{2} \int_{-\infty}^{mt} e^{s-mt} \left[\mathfrak{Y}_j(s) - \mathfrak{G}_j^{1\xi}(s) \right] ds = 0.
\end{aligned}$$

For \mathbf{I}_1 , note that (B.20) and (B.19) give

$$\begin{aligned}\mathfrak{G}_1^{1\xi}(mt) &= - \left(\int_0^1 \mathfrak{Z}_1(s) ds \right) \frac{2mt\sigma + 2mt + 1 - \sigma}{4} \\ &\quad + \int_0^{mt} \mathfrak{Z}_1(s) ds,\end{aligned}\tag{B.22}$$

$$\begin{aligned}\mathfrak{G}_1^{m\xi^m}(t) &= - \left(\int_0^m \mathfrak{Z}_1(s) ds \right) \frac{2t\sigma^m + 2t + 1 - \sigma^m}{4} \\ &\quad + \int_0^{mt} \mathfrak{Z}_1(s) ds.\end{aligned}\tag{B.23}$$

• If $\sigma = 1$ then the functions $\mathfrak{Y}_1(s)$ and $\mathfrak{Z}_1(s)$ are also 1-periodic and hence $\mathbf{I}_1 = 0$ as before.

• If $\sigma = -1$ and m is odd then $\mathfrak{Y}_1(s+1) = -\mathfrak{Y}_1(s)$ and $\mathfrak{Z}_1(s+1) = -\mathfrak{Z}_1(s)$. Note that $\int_0^m \mathfrak{Z}_1(s) ds = \int_0^1 \mathfrak{Z}_1(s) ds$ because $\int_0^2 \mathfrak{Z}_1(s) ds = 0$. By (B.22) and (B.23) we get

$$\mathfrak{G}_1^{1\xi}(mt) = \mathfrak{G}_1^{m\xi^m}(t) = -\frac{1}{2} \int_0^1 \mathfrak{Z}_1(s) ds$$

and hence $\mathbf{I}_1 = 0$.

• If $\sigma = -1$ and m is even then $\mathfrak{Y}_1(s+1) = -\mathfrak{Y}_1(s)$ and $\mathfrak{Z}_1(s+1) = -\mathfrak{Z}_1(s)$, and

$$\mathfrak{G}_1^{1\xi}(mt) = \mathfrak{G}_1^{m\xi^m}(t) - \frac{1}{2} \int_0^1 \mathfrak{Z}_1(s) ds\tag{B.24}$$

because $\int_0^m \mathfrak{Z}_1(s) ds = 0$. Then

$$\begin{aligned}\mathbf{I}_1(t) &= \frac{1}{2} \int_{mt}^{\infty} e^{mt-s} \left[\mathfrak{Y}_1(s) - \mathfrak{G}_1^{m\xi^m}(s/m) \right] ds \\ &\quad + \frac{1}{2} \int_{-\infty}^{mt} e^{s-mt} \left[\mathfrak{Y}_1(s) - \mathfrak{G}_1^{m\xi^m}(s/m) \right] ds + \mathfrak{G}_1^{m\xi^m}(t) \\ &\quad - \frac{1}{2} \int_{mt}^{\infty} e^{mt-s} \left[\mathfrak{Y}_1(s) - \mathfrak{G}_1^{1\xi}(s) \right] ds \\ &\quad - \frac{1}{2} \int_{-\infty}^{mt} e^{s-mt} \left[\mathfrak{Y}_1(s) - \mathfrak{G}_1^{1\xi}(s) \right] ds - \mathfrak{G}_1^{1\xi}(mt) \\ &= \frac{1}{2} \int_{mt}^{\infty} e^{mt-s} \left[\mathfrak{G}_1^{1\xi}(s) - \mathfrak{G}_1^{m\xi^m}(s/m) \right] ds \\ &\quad + \frac{1}{2} \int_{-\infty}^{mt} e^{s-mt} \left[\mathfrak{G}_1^{1\xi}(s) - \mathfrak{G}_1^{m\xi^m}(s/m) \right] ds \\ &\quad + \mathfrak{G}_1^{m\xi^m}(t) - \mathfrak{G}_1^{1\xi}(mt) = 0\end{aligned}$$

because of (B.24). □

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